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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Greenville, Pa., May 13.
ILLINOIS, Galesburg, May 12–13.
INDIANA, Muncie, April 28–29.
IOWA, Ames, April 21–22.
KANSAS, Topeka, April 1.
KENTUCKY, Murray, April 28–29.
LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Md., May 13.
MICHIGAN, Ann Arbor, March 18.
MINNESOTA, Northfield, May 13.
MISSOURI, Springfield, April 28.
NEBRASKA, Lincoln, May 5.
OHIO, Columbus, April 8
OKLAHOMA, Tulsa, February 10.
PHILADELPHIA, Bethlehem, Pa., December 2.
ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.
SOUTHEASTERN, Charleston, S.C., March 24–25.
SOUTHERN CALIFORNIA, Whittier, March 4.
SOUTHWESTERN, Alpine, Texas, May 2–3.
TEXAS, Abilene, March 31–April 1.
WISCONSIN, Milwaukee, May 6.

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Following a meeting held at San Francisco January 28, 1939, for the purpose of organizing a Northern California Section of the Association, By-Laws were submitted to the officers of the Association. These By-Laws have been approved by action of the Trustees, and the Northern California Section is thereby established.

W. D. CAIRNS, *Secretary-Treasurer*

LIMITS TO THE CHARACTERISTIC ROOTS OF A MATRIX*

E. T. BROWNE, University of North Carolina

1. Introduction. Let $A=(a_{rs})$ be a square matrix of order n whose elements are real or complex numbers. If I is the unit matrix and λ is a scalar indeterminate, the equation obtained by equating to zero the determinant $|A-\lambda I|$ is called the *characteristic equation* of A. The roots of this equation are called the *characteristic roots* (*latent roots*, or *characteristic values*) of A. The characteristic equation is one of the most important equations in modern algebraic analysis and has received a great deal of attention within the past seventy-five or hundred years.

If A is a matrix of some particular type, certain definite statements can be made as to the nature of its characteristic roots. For example, if A is *real symmetric*, or *Hermitian*, its characteristic roots are all *real*; if A is *real skew-symmetric*, its characteristic roots are all *pure imaginary* or zero; if A is *real orthogonal*, or *unitary*, its characteristic roots have *modulus unity*. Although it is not possible to make any definite statement as to the nature of the characteristic roots of the general matrix A, several authors have given limits to the roots. It is the purpose of this paper to list and compare the various limits that have been given and to sketch proofs of the more interesting and important of them.

2. Notations and definitions. By A' we denote the *transpose* of A; *i.e.*, the matrix obtained from A by interchanging its rows and columns. By $\bar{A}$ we denote the *conjugate* of A; *i.e.*, the matrix obtained from A by replacing each element a_{rs} by its conjugate imaginary $\bar{a}_{rs}$. Hence, $\bar{A}'$ is the *conjugate* of the *transpose* of A. By A^{-1} we denote the *inverse* of A, which exists only if A is non-singular. With these notations we say that A is *real* if $A=\bar{A}$; A is *symmetric* if $A=A'$; A is *skew-symmetric* if $A=-A'$; A is *Hermitian* if $A=\bar{A}'$; A is *orthogonal* if $A^{-1}=A'$; A is *unitary* if $A^{-1}=\bar{A}'$.

3. Bendixson's theorem.† In 1900 Bendixson proved the following theorem:

THEOREM I. *If $\alpha+i\beta$ is a characteristic root of a real n-square matrix A, then*

$$\text{(a)} \qquad |\beta| \leqq g''\left\{\frac{n(n-1)}{2}\right\}^{1/2};$$

$$\text{(b)} \qquad \rho_1 \geqq \alpha \geqq \rho_n;$$

where g'' is the largest of the quantities $|a_{rs}-a_{sr}|/2$, and $\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_n$ are the characteristic roots (all real) of the real symmetric matrix $\frac{1}{2}(A+A')$.

It will be noted that the limit furnished by the first part of this theorem is in terms of the elements of the matrix itself, while those furnished by the second part are in terms of the characteristic roots of an auxiliary matrix. Moreover, the second part is a special case of a theorem given later by Hirsch and Bromwich, and follows as an immediate consequence of a general lemma which we shall prove. We shall, therefore, confine our attention here to a proof of the first part only.

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* An address delivered by invitation before the Mathematical Association of America at Williamsburg, Virginia, December 31, 1938.

† Bendixson, Sur les racines d'une équation fondamentale, Acta Mathematica, vol. 25, 1902, pp. 359–365.

Since $\lambda=\alpha+i\beta$ is a characteristic root of A, there exists a set $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$, such that

$$\sum a_{rs}x_s = \lambda x_r, \qquad (r = 1, \cdots, n).$$

If we put $\lambda=\alpha+i\beta$, $x_s=\xi_s+i\eta_s$ multiply out and equate real parts and imaginary parts, we obtain

$$\sum_s a_{rs}\xi_s = \alpha\xi_r - \beta\eta_r, \qquad (r = 1, \cdots, n), \tag{1}$$

$$\sum_r a_{sr}\eta_r = \alpha\eta_s + \beta\xi_s, \qquad (s = 1, \cdots, n), \tag{2}$$

where in the last equation we have interchanged the subscripts r and s.

Multiply (1) through by η_r and sum as to r; similarly, multiply (2) through by ξ_s and sum as to s. On subtracting the resulting equations member for member, we obtain:

$$\beta\sum_{s=1}^{n} (\xi_s^2 + \eta_s^2) = \sum_{r,s}^{1\cdots n} (a_{sr} - a_{rs})\xi_s\eta_r.$$

Since the matrix of the bilinear form on the right is skew-symmetric, it is easy to see that this may be written:

$$\beta\sum (\xi_s^2 + \eta_s^2) = \sum_{r<s} (a_{sr} - a_{rs})(\xi_s\eta_r - \xi_r\eta_s),$$

where now the summation extends over all the $n(n-1)/2$ combinations $r<s$ of the numbers $1, \cdots, n$ taken 2 at a time. If then g'' denotes the largest of the $\frac{1}{2}|a_{sr}-a_{rs}|$, we have on substituting and squaring,

$$\beta^2\{\sum (\xi^2 + \eta^2)\}^2 \leqq 4g''^2\{\sum | \xi_s\eta_r - \xi_r\eta_s | \}^2. \tag{3}$$

Now

$$\{\sum (\xi^2 + \eta^2)\}^2 = \{\sum \xi^2 - \sum \eta^2\}^2 + 4\sum \xi^2\sum \eta^2 \geqq 4\sum \xi^2\sum \eta^2.$$

Also by Lagrange's identity,

$$\sum \xi_r^2 \sum \eta_r^2 = (\sum \xi_r\eta_r)^2 + \sum (\xi_s\eta_r - \xi_r\eta_s)^2 \geqq \sum (\xi_s\eta_r - \xi_r\eta_s)^2,$$

where the last summation extends over all the $n(n-1)/2$ combinations $r<s$. Hence

$$\{\sum (\xi^2 + \eta^2)\}^2 \geqq 4\sum_{r<s} (\xi_s\eta_r - \xi_r\eta_s)^2.$$

But by an easily established inequality for m real quantities $k_1, \cdots, k_m$,

$$\left\{ \sum_1^m k_i \right\}^2 \leqq m \sum_1^m k_i^2,$$

we have, since there are $n(n-1)/2$ terms in the summation on the right in (3),

$$\beta^2 \sum (\xi_s\eta_r - \xi_r\eta_s)^2 \leqq \frac{n(n-1)}{2} g''^2 \sum (\xi_s\eta_r - \xi_r\eta_s)^2.$$

Hence, if for at least one combination (r, s), $\xi_s\eta_r - \xi_r\eta_s \neq 0$, we have

$$|\beta| \leqq g''\left(\frac{n(n-1)}{2}\right)^{1/2}; \tag{4}$$

while if every $\xi_s\eta_r - \xi_r\eta_s = 0$, then $\beta = 0$ by (3) so that (4) still holds.

The proof of part (a) of the theorem is therefore complete. The proof of part (b) we defer until later.

In particular, if A is real symmetric, then each $a_{sr} - a_{rs} = 0$ so that $g'' = 0$, and therefore $\beta = 0$. We therefore have as a corollary the well-known theorem:

COROLLARY. *The characteristic roots of a real symmetric matrix are all real.*

The limit (4) given by Bendixson is sometimes actually attained if $n = 2$ or $n = 3$, but never (for $g'' \neq 0$) if $n > 3$, as will be seen by reference to Bromwich's and Pick's theorems which we shall state in §§4, 6. For example the limit is attained in the case of the following two matrices:

$$A = \begin{pmatrix} 0 & g'' \\ -g'' & 0 \end{pmatrix}; \qquad A = \begin{pmatrix} 0 & g'' & g'' \\ -g'' & 0 & g'' \\ -g'' & -g'' & 0 \end{pmatrix}.$$

4. Hirsch's and Bromwich's theorems. The case in which the elements of A are complex numbers was considered in 1902 by Hirsch,* and again in 1904 by Bromwich.† For convenience in formulating our results, let us write

$$b_{rs} = \frac{1}{2}(a_{rs} + \bar{a}_{sr}); \qquad c_{rs} = \frac{1}{2i}(a_{rs} - \bar{a}_{sr});$$

i.e.,

$$B = \frac{1}{2}(A + \overline{A}'); \qquad C = \frac{1}{2i}(A - \overline{A}').$$

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* Hirsch, Acta Mathematica, vol. 25, 1902, p. 367.

† Bromwich, On the roots of the characteristic equation of a linear substitution, Acta Mathematica, vol. 30, 1906, pp. 295–304.

The following note is due to Dr. E. Hellinger: O. Toeplitz, Das Algebraische Analogon zu einem Satze von Féjer, *Mathematische Zeitschrift*, vol. 2, 1918, pp. 187–197, gave the following extension of the theorem of Bendixson and Hirsch, which can be applied to forms of infinitely many variables and to integral equations: *The characteristic roots of an n-square matrix A belong to the set of all complex values, which the form* $\sum a_{rs}x_r\bar{x}_s$ *takes, if* $\sum x_r\bar{x}_r = 1$; *the boundary of this set is convex.*—E.J.M.

It is easy to verify that both B and C are Hermitian matrices.

Hirsch's theorem is as follows:

THEOREM II (a). *If $\alpha+i\beta$ is a characteristic root of an n-square matrix A, and if we denote by g the greatest of the $|a_{rs}|$, by g' the greatest of the $|b_{rs}|$, and by g'' the greatest of the $|c_{rs}|$, then*

$$|\alpha+i\beta| \leqq ng, \qquad |\alpha| \leqq ng', \qquad |\beta| \leqq ng''.$$

(b) *In the case where the a_{rs} are such that the $a_{rs}+a_{sr}$ are all real, this last inequality can be replaced by*

$$|\beta| \leqq g''\left(\frac{n(n-1)}{2}\right)^{1/2}.$$

(c) *If $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n$ are the characteristic roots of $B=\frac{1}{2}(A+\overline{A}')$, then always*

$$\rho_1 \geqq \alpha \geqq \rho_n.$$

We prove part (a) of the theorem first. Since $\lambda=\alpha+i\beta$ is a characteristic root of A, there exists a set $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ which may be normalized so that $\sum x_r\bar{x}_r=1$, and which satisfies the relations

$$\lambda x_r = \sum_{s=1}^{n} a_{rs}x_s, \qquad (r = 1, \cdots, n).$$

Multiplying these equations through by $\bar{x}_r$ and summing as to r, we have

$$\lambda = \alpha + i\beta = \sum a_{rs}\bar{x}_r x_s. \tag{5}$$

If in this last equation we take the conjugates of both members and interchange subscripts, we have

$$\bar{\lambda} = \alpha - i\beta = \sum \bar{a}_{sr}\bar{x}_r x_s;$$

whence, on adding and subtracting, we obtain:

$$\alpha = \sum b_{rs}\bar{x}_r x_s, \tag{6}$$

$$\beta = \sum c_{rs}\bar{x}_r x_s. \tag{7}$$

From the equations (5), (6) and (7) we can prove all three parts of Hirsch's Theorem II (a). Since the proofs of the three parts are practically identical, we shall prove the first part only.*

Let $\eta_r=|x_r|$ so that $\sum\eta_r^2=\sum x_r\bar{x}_r=1$. Then since $\eta_r\eta_s \leqq \frac{1}{2}(\eta_r^2+\eta_s^2)$, we have from (5)

$$|\lambda| \leqq \sum |a_{rs}||\bar{x}_r||x_s| \leqq \tfrac{1}{2}g\sum_{rs}(\eta_r^2+\eta_s^2)$$

$$= \tfrac{1}{2}g\sum_{s=1}^{n}\sum_{r=1}^{n}\eta_r^2 + \tfrac{1}{2}g\sum_{r=1}^{n}\sum_{s=1}^{n}\eta_s^2 = \tfrac{1}{2}ng + \tfrac{1}{2}ng = ng.$$

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* In fact, the last two parts of Theorem II (a) follow as direct consequences of the first part and of Theorems II (c) and III.

This limit for $|\lambda|$ is actually attained if A is a *real* matrix every one of whose elements is equal to the same number g; and indeed this limit is not attained in any other case.

The proof of Hirsch's Theorem II (b) is quite similar to that of Theorem I (a) as given by Bendixson.

Before proceeding to a proof of Theorem II (c), we shall establish a lemma, which will be useful also in proving Bromwich's theorem and Theorem VII.

LEMMA I. *If* $H=(h_{rs})$ *is an Hermitian matrix with characteristic roots* $R_1 \geqq R_2 \geqq \cdots \geqq R_n$, *and if the equality*

$$\sum h_{rs}\bar{x}_r x_s = \sigma \sum x_r \bar{x}_r \tag{8}$$

is satisfied by a set $(x) \neq (0)$, *then*

$$R_1 \geqq \sigma \geqq R_n,$$

The proof of the lemma is very simple. For by a conjunctive transformation $x_r = \sum u_{rs} y_s$, where the matrix $U=(u_{rs})$ is unitary, the equality (8) is transformed into*

$$R_1 y_1 \bar{y}_1 + R_2 y_2 \bar{y}_2 + \cdots + R_n y_n \bar{y}_n = \sigma \sum y_r \bar{y}_r.$$

Hence, if R_1 is the greatest and R_n the smallest of the R's, we have since each $y_r \bar{y}_r \geqq 0$,

$$R_1 \sum y_r \bar{y}_r \geqq \sum R_r y_r \bar{y}_r = \sigma \sum y_r \bar{y}_r \geqq R_n \sum y_r \bar{y}_r,$$

so that

$$R_1 \geqq \sigma \geqq R_n.$$

Now let us return to equations (6) and (7). Since both the matrices B and C are Hermitian, it follows immediately from the lemma that if $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n$ are the characteristic roots of B while $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n$ are the characteristic roots of C, then

$$\rho_1 \geqq \alpha \geqq \rho_n,$$

$$\mu_1 \geqq \beta \geqq \mu_n.$$

We have, therefore, established not only Hirsch's Theorem II (c) but also the following theorem due to Bromwich:

THEOREM III. *If* $\alpha + i\beta$ *is a characteristic root of* A *and if* $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n$ *are the characteristic roots* (*all real*) *of the Hermitian matrix* $C=(A-\bar{A}')/(2i)$, *then*

$$\mu_1 \geqq \beta \geqq \mu_n. \tag{9}$$

In particular, if A is a real matrix so that $K = iC = \frac{1}{2}(A - A')$ is real skew-symmetric with non-zero characteristic roots

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* Turnbull and Aitken, Theory of Canonical Matrices, London, 1932, p. 106.

$$\pm i\mu_1,\ \pm i\mu_2, \cdots, \pm i\mu_t \qquad (2t \leqq n), \tag{10}$$

Bromwich shows that the limit $|\beta| \leqq |\mu|$, where $|\mu|$ is the greatest of the $|\mu_r|$, is more restricted than that given by Bendixson. For the sum of the products two at a time of the roots in (10) is easily seen to be

$$\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2.$$

But this is equal to the coefficient of λ^{n-2} in the expansion of $|\lambda I - K|$; *i.e.*,

$$\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2 = \sum_{r<s} \begin{vmatrix} 0 & k_{rs} \\ -k_{rs} & 0 \end{vmatrix} = \sum_{r<s} k_{rs}^2,$$

where the summation extends over the $n(n-1)/2$ combinations $r<s$ taken two at a time. If then g'' is the largest of the $|k_{rs}|$ and $|\mu|$ is the largest of the $|\mu_r|$, we have

$$|\mu|^2 \leqq \sum |\mu_r|^2 = \sum k_{rs}^2 \leqq \frac{n(n-1)}{2} g''^2,$$

so that

$$|\mu| \leqq g''\left(\frac{n(n-1)}{2}\right)^{1/2}.$$

Thus, the limit on the imaginary part of the characteristic root of a real matrix A as given by Bromwich is (in general) more restricted than that given earlier by Bendixson. However, the limit given by the former is in terms of the characteristic roots of an auxiliary matrix, while the limit given by the latter is in terms of the elements of the matrix itself.

5. Schur's proofs. In 1909 I. Schur* gave a demonstration of Hirsch's Theorem II (a) and of Bendixson's Theorem I (a) which seems to be of sufficient interest to be included here. Schur first establishes the following lemma:

LEMMA II. *If U is any unitary matrix and if $\bar{U}'AU = T$, then*

$$\sum_{r,s}^{1\cdots n} |t_{rs}|^2 = \sum_{r,s}^{1\cdots n} |a_{rs}|^2;$$

that is, if two matrices A and T are unitary equivalent, the sum of the squares of the absolute values of all the elements of A is equal to the corresponding sum for T.

For from

$$\bar{U}'AU = T, \tag{11}$$

we have on taking conjugate transposes,

$$\bar{U}'\bar{A}'U = \bar{T}'; \tag{12}$$

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* I. Schur, Über die characteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen, Mathematische Annalen, vol. 65, 1909, pp. 488–510.

whence

$$T\overline{T}' = \bar{U}'AU\bar{U}'\overline{A}'U = U^{-1}A\overline{A}'U, \quad \text{since} \quad \bar{U}' = U^{-1}.$$

Hence, the matrices $T\overline{T}'$ and $A\overline{A}'$ are *similar*, and therefore have the same trace.* But the trace of $T\overline{T}'$ is seen to be precisely

$$\sum_{r,s}^{1,\cdots,n} t_{rs}\bar{t}_{rs} = \sum_{r,s}^{1,\cdots,n} |\, t_{rs}\,|^2.$$

The lemma is therefore established.

Now let the characteristic roots of A be

$$\lambda_r = \alpha_r + i\beta_r, \qquad (r = 1, \cdots, n).$$

Then by an easily established and well-known theorem,† we know that A is unitary equivalent to a *triangular* matrix of the form

$$T = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots 0 & 0 \\ t_{21} & \lambda_2 & 0 & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ t_{n-1\,1} & t_{n-1\,2} & t_{n-1\,3} & \cdots \lambda_{n-1} & 0 \\ t_{n\,1} & t_{n\,2} & t_{n\,3} & \cdots t_{n\,n-1} & \lambda_n \end{pmatrix}.$$

Hence, by the lemma, we have

$$\sum |\, \lambda_r\,|^2 \leqq \sum |\, \lambda_r\,|^2 + \sum_{r>s} |\, t_{rs}\,|^2 = \sum |\, a_{rs}\,|^2. \tag{13}$$

If then $|\lambda|$ is the greatest of the $|\lambda_r|$ and g is the greatest of the $|a_{rs}|$, we have at once

$$|\, \lambda\,|^2 \leqq \sum |\, \lambda_r\,|^2 \leqq n^2g^2,$$

whence

$$|\, \lambda\,| \leqq ng.$$

It is clear that the equality sign holds in (13) if, and only if, U can be so chosen that $t_{rs} = 0$, $r \neq s$, that is, A is unitary equivalent to a diagonal matrix or is *normal*.

Moreover, with T in the form above, we have on combining (11) and (12),

$$\bar{U}'\left(\frac{A \pm \overline{A}'}{2}\right)U = \frac{T \pm \overline{T}'}{2}, \tag{14}$$

where the diagonal elements of the matrix on the right are in the respective cases $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $i\beta_1, i\beta_2, \cdots, i\beta_n$, the absolute values of the remaining elements being $|t_{rs}|/2$. Hence by the lemma we have

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* By the *trace* of a matrix we mean the sum of the elements in the principal diagonal of the matrix. It is invariant under *similarity* transformations. Cf. MacDuffee, The Theory of Matrices Berlin, 1933, p. 19; also p. 69.

† MacDuffee, *loc. cit.*, p. 75.

$$\sum |\alpha_r|^2 \leqq \sum |\alpha_r|^2 + \sum \left|\frac{t_{rs}}{2}\right|^2 = \sum |b_{rs}|^2, \tag{15}$$

$$\sum |\beta_r|^2 \leqq \sum |\beta_r|^2 + \sum \left|\frac{t_{rs}}{2}\right|^2 = \sum |c_{rs}|^2, \tag{16}$$

where b_{rs} and c_{rs} have the meaning given in section **4**. If then we denote by g' and g'', respectively, the greatest of the $|b_{rs}|$ and $|c_{rs}|$, we have immediately

$$|\alpha_r| \leqq ng', \qquad |\beta_r| \leqq ng''.$$

But further, if A is *real*, then in (16) $c_{rr}=0$, so that

$$\sum |\beta_r|^2 \leqq n(n-1)g''^2;$$

and since complex roots occur in pairs so that each $\beta \neq 0$ occurs at least twice, we have Bendixson's result

$$|\beta_r| \leqq g''\left(\frac{n(n-1)}{2}\right)^{1/2}.$$

6. Pick's Theorem. In 1922 Pick* gave a proof of Hirsch's Theorem II (a) and II (c), and in addition for a *real* matrix A gave a limit for $|\beta|$ which is in general more restricted than that given by Bendixson. Pick's theorem is as follows:

THEOREM V. *If $\alpha + i\beta$ is a characteristic root of a real n-square matrix A and if g'' is the largest of the $|k_{rs}| = |a_{rs} - a_{sr}|/2$ then*

$$|\beta| \leqq g'' \cot \frac{\pi}{2n}.$$

To prove Pick's theorem we recall that the characteristic roots of $K = \frac{1}{2}(A - A')$ are all pure imaginary or zero. If $|\mu|$ is the greatest of the absolute values of the roots, then since by Bromwich's Theorem $|\beta| \leqq |\mu|$, we need only show that

$$|\mu| \leqq g'' \cot \frac{\pi}{2n}.$$

Since $i\mu$ is a characteristic root of K, there exists a set $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$ such that

$$i\mu x_r = \sum k_{rs} x_s,$$

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* Pick, Georg, Über die Wurzeln der characteristischen Gleichungen von Schwingungsproblemen, Zeitschrift für angewandte Mathematik und Mechanik, vol. 2, 1922, pp. 353–357.

whence

$$i\mu \sum x_r \bar{x}_r = \sum_{r,s}^{1,\cdots,n} k_{rs} \bar{x}_r x_s. \tag{17}$$

If any one of the numbers x_r has an amplitude θ_r such that $\pi/2 < \theta_r \leqq 3\pi/2$ we can change its sign thus obtaining a set such that each x_r has amplitude θ_r where $-\pi/2 < \theta_r \leqq \pi/2$. We now apply, if necessary, a permutation to the x's so that the resulting set has the property that for $r < s$ we have $\theta_r \leqq \theta_s$.

These last two changes can be looked upon as real orthogonal transformations on the x's and every such transformation leaves the left member of (17) unaltered, while it transforms the right member into an expression of the same type, where the k's differ at most in sign or order from the original k's. We may then write (17)

$$\mu \sum x_r \bar{x}_r = \sum_{r<s} k_{rs} \left(\frac{\bar{x}_r x_s - \bar{x}_s x_r}{i} \right)$$

where the summation on the right extends to all the $n(n-1)/2$ combinations $(r < s)$ and where $\theta_r \leqq \theta_s$ for $r < s$. It is easy to see that the expression in the parentheses is never negative. Hence, if g'' is the greatest of the $|k_{rs}|$, we have

$$|\mu| \sum x_r \bar{x}_r \leqq g'' \sum_{r<s} \left(\frac{\bar{x}_r x_s - \bar{x}_s x_r}{i} \right);$$

i.e.,

$$\frac{|\mu|}{g''} \sum x_r \bar{x}_r \leqq \frac{1}{i} \sum_{r,s}^{1,\cdots,n} d_{rs} \bar{x}_r x_s,$$

where $D = (d_{rs})$ is the real skew-symmetric matrix with $d_{rs} = 1$ for $r < s$ and $d_{rs} = -1$ for $r > s$. Since D/i is Hermitian, by Lemma I, $|\mu|/g''$ cannot exceed in absolute the greatest characteristic root of D. The characteristic equation of D is

$$\begin{vmatrix} -\lambda, & 1, & 1, \cdots, & 1, & 1 \\ -1, & -\lambda, & 1, \cdots, & 1, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -1, & -1, & -1, \cdots, & -\lambda, & 1 \\ -1, & -1, & -1, \cdots, & -1, & -\lambda \end{vmatrix} = 0,$$

which on expansion gives $\sum_{j=0}^{n} \sigma_j (-\lambda)^{n-j} = 0$, where σ_j is the sum of all the j-rowed principal minor determinants Δ_j of D. Every principal minor determinant Δ_{2s+1} of odd order is zero, while each Δ_{2s} of even order has the value $+1$. This last statement is easily established by induction. For it is obviously true if $2s = 2$. Moreover, if in Δ_{2s} we add the last column to the first and in the resulting determinant add the first row to the last, we have $\Delta_{2s} = \Delta_{2s-2}$, so that $\Delta_{2s} = 1$.

Since there are exactly $\binom{n}{j}$ principal minor determinants Δ_j of order j in D, the characteristic equation of D can therefore be written

$$(-1)^n\left\{\lambda^n + \binom{n}{2}\lambda^{n-2} + \binom{n}{4}\lambda^{n-4} + \cdots\right\} = 0,$$

where the final term on the left is 1 or $n\lambda$ according as n is even or odd. But this is seen to be precisely the same as*

$$\tfrac{1}{2}\{(\lambda+1)^n + (\lambda-1)^n\} = 0. \tag{18}$$

On solving (18) we obtain

$$\lambda = \frac{\alpha+1}{\alpha-1}, \quad \text{where} \quad \alpha^n = -1; \tag{19}$$

i.e.,

$$\alpha = \cos\frac{(2k+1)\pi}{n} + i\sin\frac{(2k+1)\pi}{n}, \qquad (k = 0, 1, \cdots, n-1).$$

From (19) it is easy to show that

$$\lambda = -i\cot\frac{(2k+1)\pi}{2n}, \qquad (k = 0, 1, \cdots, n-1).$$

Now the largest of these roots in absolute value is $\cot \pi/2n$. Hence, we have

$$|\mu| \leqq g'' \cot\frac{\pi}{2n}.$$

This limit is in general more restricted than that given by Bendixson. For the roots of (18) are

$$\lambda_1 = -i\cot\frac{\pi}{2n}, \ \lambda_2 = -i\cot\frac{3\pi}{2n}, \cdots, \lambda_n = -i\cot\frac{(2n-1)\pi}{2n}.$$

Moreover, from (18) we have

$$\sum \lambda_r = 0, \qquad \sum \lambda_r\lambda_s = \frac{n(n-1)}{2},$$

whence

$$\sum \lambda_r^2 = -n(n-1).$$

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* Cf. Hölder, Leipziger Berichte, vol. 65, pp. 110–120; also vol. 66, pp. 98–102. Cf. also Hirsch, Mathematische Annalen, vol. 52, pp. 150 ff.

that is,

$$\cot^2\frac{\pi}{2n}+\cot^2\frac{3\pi}{2n}+\cdots+\cot^2\frac{(2n-1)\pi}{2n}=n(n-1).$$

But

$$\cot^2\frac{\pi}{2n}=\cot^2\frac{(2n-1)\pi}{2n},$$

whence

$$\cot^2\frac{\pi}{2n}\leqq\frac{n(n-1)}{2}\cdot$$

7. More recent limits to the roots. In 1930 the author* gave a refinement of Hirsch's Theorem II (a) which was further refined by Parker† in 1937.

Returning to equations (5) let us write as before $|x_r|=\eta_r$, so that $\sum\eta_r^2=1$. We then have

$$|\lambda|\leqq\sum|a_{rs}|\eta_r\eta_s\leqq\tfrac{1}{2}\sum_{r,s}^{1,\cdots,n}|a_{rs}|(\eta_r^2+\eta_s^2)$$

$$=\tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}|a_{rs}|\eta_r^2+\tfrac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n}|a_{rs}|\eta_s^2.$$

Let us now denote by σ_j (τ_j) the sum of the absolute values of the elements in the jth row (column) of A; *i.e.*, let

$$\sigma_j=\sum_{s=1}^{n}|a_{js}|,\qquad\tau_j=\sum_{r=1}^{n}|a_{rj}|,$$

and let σ (τ) be the greatest of the σ_j (τ_j). We then have

$$|\lambda|\leqq\tfrac{1}{2}\sum_r\sigma_r\eta_r^2+\tfrac{1}{2}\sum_s\tau_s\eta_s^2=\tfrac{1}{2}\sum(\sigma_r+\tau_r)\eta_r^2$$

$$\leqq\tfrac{1}{2}(\sigma+\tau)\sum\eta_r^2=\tfrac{1}{2}(\sigma+\tau).$$

In a similar manner it follows from (6) and (7) that if σ' is the greatest of the $\sigma_j'=\sum|b_{jr}|=\sum|b_{rj}|$, and if σ'' is the greatest of the $\sigma_j''=\sum|c_{jr}|=\sum|c_{rj}|$, then

$$|\alpha|\leqq\sigma',\qquad|\beta|\leqq\sigma''.$$

It is clear that these limits for $|\lambda|$, $|\alpha|$, $|\beta|$ are never greater, and are in general less, than those given by Hirsch.

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* Browne, The characteristic roots of a matrix, Bulletin of the American Mathematical Society, vol. 36, 1930, pp. 705–710.

† Parker, The characteristic roots of a matrix, Duke Mathematical Journal, vol. 3, 1937, pp. 484–487.

THEOREM VI. *If* $\lambda=\alpha+i\beta$ *is a characteristic root of an* n*-square matrix* A, *and if we denote by* σ, τ, σ', σ'' *respectively, the greatest of the*

$$\sigma_j = \sum_s |a_{js}|, \qquad \tau_j = \sum_r |a_{rj}|,$$

$$\sigma_j' = \sum_r \tfrac{1}{2}|a_{jr} + \bar{a}_{rj}|, \qquad \sigma_j'' = \sum_r \tfrac{1}{2}|a_{rj} - \bar{a}_{jr}|,$$

then

$$|\lambda| \leqq \tfrac{1}{2}(\sigma + \tau), \qquad |\alpha| \leqq \sigma', \qquad |\beta| \leqq \sigma''.$$

8. Limits in terms of the matrix $A\bar{A}'$. The limits to the characteristic roots of a matrix A which we have given up to this point have been either in terms of the elements of the matrix itself or in terms of the characteristic roots of the auxiliary matrices $B=\frac{1}{2}(A+\bar{A}')$ and $C=(A-\bar{A}')/(2i)$. We shall now give a limit in terms of the characteristic roots of the positive definite matrix $A\bar{A}'$, which in some cases, notably in the case of a *unitary* matrix, or of a *real orthogonal* matrix, gives limits which are more restricted than those given earlier.

The matrix $A\bar{A}'=M$ is clearly Hermitian; i.e., $\bar{M}'=M$. Moreover, its characteristic roots are all $\geqq 0$ and are the same as the characteristic roots of $\bar{A}'A$.* Suppose then that λ is a characteristic root of A, so that there exists a set $(x)\neq(0)$ such that

$$\sum_t a_{ts}x_t = \lambda x_s, \qquad (s = 1, \cdots, n).$$

Taking conjugates of both members, we have

$$\sum_r \bar{a}_{rs}\bar{x}_r = \bar{\lambda}\bar{x}_s.$$

Multiplying these two equations, member for member, and summing as to s, we have

$$\sum_{r,t}^{1,\cdots,n}\left(\sum_s a_{ts}\bar{a}_{rs}\right)x_t\bar{x}_r = \lambda\bar{\lambda}\sum x_s\bar{x}_s.$$

Since the matrix of the Hermitian form on the left is $A\bar{A}'$, it follows from the lemma of section **4** that $\lambda\bar{\lambda}$ lies between the greatest and the least of the characteristic roots of $A\bar{A}'$. We have then the following theorem:

THEOREM VII. *If* λ *is a characteristic root of a matrix* A, *and if* $R_1\geqq R_2\geqq \cdots \geqq R_n$ *are the characteristic roots* (*all* $\geqq 0$) *of* $A\bar{A}'$, *then*

$$R_1 \geqq \lambda\bar{\lambda} \geqq R_n.$$

By this theorem the characteristic roots of a matrix are restricted to an annular region around the origin in the complex plane, with inner and outer radii $R_n^{1/2}$ and $R_1^{1/2}$, respectively. This region of course enlarges to a circle if A is singular.

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* Browne, The characteristic equation of a matrix, Bulletin of the American Mathematical Society, vol. 34, 1928, pp. 363–368.

In the particular case in which A is unitary, or real orthogonal, we have $A\overline{A}'=I$, so that $R_1=R_n=1$. In this case the annular region shrinks to the periphery of a circle of radius unity. As a corollary to Theorem VII we therefore have the well known result:

COROLLARY. *The characteristic roots of a unitary matrix are of modulus unity.*

9. Alternate proofs of Theorems II(c), III and VII. The proofs of Theorems II (c), III and VII which we have given were all based on Lemma I concerning an Hermitian matrix. Alternate proofs of these theorems can be given which are so simple as to justify their inclusion here. These proofs are based on the following lemma:*

LEMMA III. *If $R_1 \geqq R_2 \geqq \cdots \geqq R_n$ are the characteristic roots of the Hermitian matrix $H=(h_{rs})$, then*

$$R_1 \geqq h_{rr} \geqq R_n;$$

that is, the matrix has at least one characteristic root which is not less than the greatest diagonal element, and at least one characteristic root which is not greater than the least diagonal element.

The proof of this lemma is based on the well known fact that every diagonal element of a positive definite Hermitian matrix is $\geqq 0$.

First, suppose that $h_{kk}>R_1$. If then σ is a real number such that $h_{kk}>\sigma>R_1$, the form

$$\sigma \sum \bar{x}_r x_r - \sum h_{rs}\bar{x}_r x_s$$

is positive definite. This follows directly from the argument employed in the proof of Lemma I. But at least one diagonal element $\sigma-h_{kk}$ is negative, which is impossible.

Similarly, if $h_{kk}<R_n$ and if $h_{kk}<\sigma<R_n$, the form

$$\sum h_{rs}\bar{x}_r x_s - \sigma \sum \bar{x}_r,\ x_r$$

is positive definite, while at least one diagonal element in its matrix is <0.

Now, to apply this lemma to the proof of our theorems, we choose the unitary matrix U such that $\bar{U}'AU=T$, where T is triangular as in section 5. We then have as in (14)

$$\bar{U}'\left(\frac{A+\overline{A}'}{2}\right)U = \frac{T+\overline{T}'}{2}; \tag{14'}$$

$$\bar{U}'\left(\frac{A-\overline{A}'}{2i}\right)U = \frac{T-\overline{T}'}{2i}. \tag{14''}$$

The matrix $(T+\overline{T}')/2$ is Hermitian with diagonal elements $\alpha_1, \cdots, \alpha_n$ where the notation is that of section 5. If then $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n$ are the characteristic roots of $(T+\overline{T}')/2$, and therefore of $(A+\overline{A}')/2$, we have by the lemma:

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* *Cf.* Parker, Duke Mathematical Journal, *op. cit.*, Theorem 2, p. 487.

$$\rho_1 \geqq \alpha_r \geqq \rho_n, \qquad (r = 1, 2, \cdots, n).$$

Similarly, the diagonal elements of $(T-\overline{T}')/2i$ are $\beta_1, \cdots, \beta_n$. If the characteristic roots in this case are $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n$, then

$$\mu_1 \geqq \beta_r \geqq \mu_n.$$

Thus we have established Theorems II (c) and III.

Moreover from $\bar{U}'A\overline{A}'U = T\overline{T}'$, we have, since the element in the first row and the first column of $T\overline{T}'$ is $\lambda_1\bar{\lambda}_1$,

$$R_1 \geqq \lambda_1\bar{\lambda}_1 \geqq R_n,$$

where $R_1 \geqq R_2 \geqq \cdots \geqq R_n \geqq 0$ are the characteristic roots of $A\overline{A}'$. Since in the triangular form T any one of the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ may be taken as λ_1, this furnishes an alternate proof of Theorem VII.

ANALYTIC TREATMENT OF SOME ORTHOPOLE THEOREMS

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1. Introduction. In recent years a considerable number of writers have made effective use of complex coördinates in proving geometrical theorems. Last year J. R. Musselman published in this MONTHLY[*] an article using this method in treating the orthopole of a line with respect to a triangle.[†] It is the object of the present paper to extend the discussion concerning the orthopole by this analytic method, particularly to give results obtained by other methods by Neuberg, Lemoyne, Lienard, and Gallatly.

We may first note that F. D. Murnaghan gave a treatment of the orthopole theorem and related properties by analytic methods in *Mathesis* in 1927 (pp. 24–28), and that R. Bouvaist gave extensions to the isopole in the same volume (pp. 97–101; 211–217; 355–359) by the same methods.

2. Orthopoles with respect to parallel lines. Let $A_1A_2A_3$ be a triangle inscribed in the base-circle Γ, having as center O, the *turns*[‡] corresponding to the vertices being t_1, t_2, t_3 and the unit-point Ω corresponding to the turn 1; the symmetric functions of t_1, t_2, t_3 are

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[*] On the line of images, vol. 45, 1938, pp. 421–430. For an extended treatment of orthopole-geometry, see my paper in the Tôhoku Mathematical Journal, 1926–27, pp. 77–125.

[†] The orthopole of a line Δ with respect to a triangle $A_1A_2A_3$ may be described as follows. Let B_i be the foot of the perpendicular from A_i to Δ. Through B_i draw a line perpendicular to the side opposite A_i; the three lines thus obtained by taking $i=1, 2, 3$ meet in a point called the orthopole of Δ with respect to $A_1A_2A_3$.

[‡] Complex numbers whose moduli are unity, see Frank and F. V. Morley, Inversive Geometry, 1933, p. 15.

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3.$$

When the equation of any straight line is written in its self-conjugate form

$$x/a + \bar{x}/\bar{a} = 1,$$

where $\bar{x}$ is the conjugate of x, and $\bar{a}$ the conjugate of a, the directed distance from the point y to that line is

$$|a|(1 - y/a - \bar{y}/\bar{a})/2.$$

Let then Δ be a straight line perpendicular to $O\Omega$, the angle $(O\Omega, \Delta)$ being a right angle in the positive sense.

If 2δ is the directed distance from O to Δ, the equation of Δ is

$$x + \bar{x} = 2\delta, \qquad (\delta \text{ real}).$$

The projection of A_1 on Δ is the point

$$\delta + (t_1^2 - 1)/2t_1,$$

and the perpendicular from this point on A_2A_3 is

$$2(xt_1 - \bar{x}s_3) = 2\delta t_1(1 - t_2t_3) + (t_1^2 - 1)(1 + t_2t_3).$$

This line and the two others obtained by starting with A_2 and A_3 instead of A_1 meet at the point

$$\omega = \delta + (s_1 + s_3)/2,$$

the orthopole of Δ as to the triangle $A_1A_2A_3$.

The directed distance from ω to Δ is

$$d = -(s_1 + s_3 + \bar{s}_1 + \bar{s}_3)/4.$$

For all parallel lines the distance from these lines to their respective orthopoles is constant.

Since $\bar{\omega} = \delta + (s_2 + 1)/2s_3$,

$$\omega - \bar{\omega} = (s_1s_3 + s_3^2 - s_2 - 1)/2s_3.$$

Hence we have a theorem due to Soons,* *the orthopoles of a system of parallel lines lie on a line perpendicular to the system.*

3. The orthopole and Simson lines. The Simson line of a point τ on Γ is

$$x\tau - \bar{x}s_3 = (\tau^3 + s_1\tau^2 - s_2\tau - s_3)/2\tau,$$

and, since

$$\bar{\omega} = \delta + (s_2 + 1)/2s_3,$$

the condition for ω to be on the Simson line of τ may be easily transformed into

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* *Mathesis*, 1896, p. 57.

$$(\tau - s_3)(\tau^2 - 2\delta\tau + 1) = 0.$$

When $\tau = s_3$, the Simson line is perpendicular to Δ; the second factor shows that ω is at the intersection of the Simson lines of the points τ and τ' such that

$$\tau + \tau' = 2\delta, \qquad \tau\tau' = 1,$$

i.e., at the intersection of the Simson lines of the points where Δ cuts Γ.

The orthopole ω is at the intersection of the Simson line perpendicular to Δ with the Simson lines of the intersections of Δ with the circumcircle.

4. The power of the orthopole. Call α a point on Δ and κ the power of α with respect to Γ. The counter-point β to α in the triangle $A_1A_2A_3$ is then such that*

$$-(1 - \bar{\alpha}\beta)\kappa = 1 - s_1\bar{\alpha} + s_2\bar{\alpha}^2 - s_3\bar{\alpha}^3,$$

and the center Q of the common pedal circle γ of α and β is

$$-\left[s_1 - (s_2 + \alpha^2)\bar{\alpha} + s_3\bar{\alpha}^2\right]/2\kappa.$$

Hence, as $\alpha + \bar{\alpha} = 2\delta$,

$$\begin{aligned} 4s_3\kappa^2\overline{Q\omega}^2 = {} & \left[-s_1\alpha\bar{\alpha} + s_2\bar{\alpha} - s_3(\kappa + \bar{\alpha}^2) - 2\delta\kappa + \alpha^2\bar{\alpha}\right] \\ & \times \left[s_1\alpha - s_2\alpha\bar{\alpha} - s_3(2\delta\kappa - \alpha\bar{\alpha}^2) - \kappa - \alpha^2\right]. \end{aligned}$$

But the radius ρ of γ is given by†

$$4\rho^2 = (1 - \bar{\alpha}\beta)(1 - \alpha\bar{\beta})$$

or‡

$$4s_3\kappa^2\rho^2 = (1 - s_1\bar{\alpha} + s_2\bar{\alpha}^2 - s_3\bar{\alpha}^3)(s_3 - s_2\alpha + s_1\alpha^2 - \alpha^3).$$

Hence

$$\overline{Q\omega}^2 - \rho^2 = \delta(s_1s_3 + s_2 + s_3^2 + 1)/2s_3 = -2d\delta.$$

The power of the orthopole ω for the pedal circle of any point on Δ is twice the product of the distance from ω to Δ by the distance from Δ to the circumcenter.§

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* Frank and F. V. Morley, *loc. cit.*, p. 196; "counter-points" are also called "isogonal conjugate points" or "focal pairs."

† F. and F. V. Morley, *loc. cit.*, p. 198.

‡ The formula may also be written

$$4\kappa^2\rho^2 = \Pi(\alpha - t_i)(\bar{\alpha} - \bar{t}_i) = \lambda_1^2\lambda_2^2\lambda_3^2,$$

λ_1, λ_2, λ_3 being the distances from α to A_1, A_2, A_3 and we obtain a theorem given by Neuberg in Triangles et coniques combinés, Nouvelles Annales de Mathématiques, 1870, p. 65.

§ Gallatly, The Modern Geometry of the Triangle, 2nd ed., p. 51.

5. Other theorems. Let us determine the reciprocal transversal to

$$x + \bar{x} = h,$$

a parallel Δ' to Δ. The intersection with A_2A_3 being

$$(t_2 + t_3 - ht_2t_3)/(1 - t_2t_3)$$

the image of this intersection in the mid-point of A_2A_3 is

$$\xi_1 = t_2t_3(t_2 + t_3 - h)/(t_2t_3 - 1),$$

and the straight line Δ_0' (reciprocal to Δ') passing through ξ_1 and the two similar points is

$$x(\bar{s}_1 - \bar{s}_3 - h) + \bar{x}(s_1 - s_3 - h) = s_1\bar{s}_1 - h(s_1 + \bar{s}_1) + h^2 - 1.$$

When $h = 2\delta$, *i.e.*, when $\Delta' \equiv \Delta$, the equation of Δ_0' is

$$x(\bar{s}_1 - \bar{s}_3 - 2\delta) + \bar{x}(s_1 - s_3 - 2\delta) = s_1\bar{s}_1 - 2\delta(s_1 + \bar{s}_1) + 4\delta^2 - 1,$$

a straight line perpendicular to

$$x(\bar{s}_1 - \bar{s}_3 - 2\delta) - \bar{x}(s_1 - s_3 - 2\delta) = s_2 - \bar{s}_2 - 2\delta(s_1 - \bar{s}_1),$$

the join of the orthocenter s_1 to the orthopole ω.

The orthopole of a line Δ lies on the perpendicular from the orthocenter on the reciprocal line to Δ.[*]

When $h = -2\delta$ the equation of Δ_0' becomes

$$x(\bar{s}_1 - \bar{s}_3 + 2\delta) + \bar{x}(s_1 - s_3 + 2\delta) = s_1\bar{s}_1 + 2\delta(s_1 + \bar{s}_1) + 4\delta^2 - 1,$$

and Δ_0' passes through ω.

The orthopole of a line Δ lies on the reciprocal to the image of Δ in the circumcenter.

When Δ is a Simson line, $\tau = -s_3$ and ω is the point

$$\omega = (3s_1 + \bar{s}_1 + s_3 - \bar{s}_3)/4;$$

as, in this case

$$\bar{\omega} = (s_1 + 3\bar{s}_1 - s_3 + \bar{s}_3)/4,$$

ω is on the line

$$x + \bar{x} = s_1 + \bar{s}_1,$$

i.e., the parallel to Δ through the orthocenter s_1.

As the orthopole lies also on the Simson line, perpendicular to Δ, we have the following theorem:

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[*] R. Goormaghtigh, Nouvelles Annales de Mathématiques, 1919, p. 39; Opperman, Le quadrilatère complet, 2nd ed., p. 69; de Lepiney, Mathesis, 1922, p. 314; Servais, Mathesis, 1923, p. 152.

The orthopole of the Simson line of a point M on the circumcircle is the projection of the orthocenter on the Simson line of the image of M in the circumcenter.[*]

When $\delta = 0$, Δ is a circumdiameter and

$$2\omega = s_1 + s_3,$$

a relation which shows a very simple geometric meaning of the symmetric function s_3: *the point s_3 is the image of the orthocenter in the orthopole of the circumdiameter perpendicular to the base line.*

A SPECIAL PAIR OF ISOTOMIC CONJUGATES[†]

HERBERT WOLFE, Brooklyn College

1. Introduction. With any triangle $A_1A_2A_3$ there are associated numerous special points possessing interesting properties. In this paper we shall discuss two points α and α', defined as follows:

$$A_1\alpha_3 = A_2\alpha_1 = A_3\alpha_2, \qquad A_1\alpha_2' = A_2\alpha_3' = A_3\alpha_1',$$

where α_i (α_i') is the point of intersection of $A_i\alpha$ ($A_i\alpha'$) and the side opposite A_i.[‡]

2. Elementary Theorems.

THEOREM 1. *In any triangle there is one and only one pair of points α and α'.*

Proof. Let

$$A_1\alpha_3 = A_2\alpha_1 = A_3\alpha_2 = x.$$

Then, by Ceva's Theorem,

$$x^3 = (a_1 - x)(a_2 - x)(a_3 - x) = S_3 - S_2x + S_1x^2 - x^3,$$

where

$$a_1 = A_2A_3, \qquad a_2 = A_3A_1, \qquad a_3 = A_1A_2.$$

$$S_1 = a_1 + a_2 + a_3, \qquad S_2 = a_1a_2 + a_2a_3 + a_3a_1, \qquad S_3 = a_1a_2a_3.$$

We see that

$$2x^3 - S_1x^2 + S_2x - S_3 = 0.$$

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[*] R. Goormaghtigh, Mathesis, 1914, p. 152. This is also a special case of one of the theorems in section **5**, as any Simson line is perpendicular to its reciprocal.

[†] The theorems in this paper were developed in a course in Modern Geometry given by Prof H. F. MacNeish at Brooklyn College.

[‡] R. A. Johnson, Modern Geometry, p. 157.

A somewhat similar situation is discussed by R. Goormaghtigh in the article entitled "Sur deux points du plan d'un triangle et sur une géneralization des points de Brocard"; Nouvelles Annales de Mathématique, vol. 18 (1918), pp. 417–424. He discusses two points B and B', whose projections on the sides A_2A_3, A_3A_1, A_1A_2 of triangle $A_1A_2A_3$ are B_1, B_2, B_3 and B_1', B_2', B_3', respectively, where $A_1B_3 = B_3'A_2 = A_2B_1 = B_1'A_3 = A_3B_2 = B_2'A_1$.

If the roots of this equation be decreased by $S_1/6$, the resultant equation is

$$x^3 + (6S_2 - S_1^2)x/12 + (9S_1S_2 - S_1^3 - 54S_3)/108 = 0.$$

If the discriminant, Δ, of this equation is negative, then there exists one and only one real root.* The equation $x^3+px+q=0$ has for its discriminant, $\Delta \equiv -4p^3-27q^2$. Therefore, if $p>0$, then $\Delta<0$. It remains to be shown that, $6S_2-S_1^2>0$.

In a triangle having sides a_1, a_2, a_3,

$$a_1^2 - 2a_1a_2 + a_2^2 < a_3^2\,,$$
$$a_2^2 - 2a_2a_3 + a_3^2 < a_1^2\,,$$
$$a_3^2 - 2a_3a_1 + a_1^2 < a_2^2\,.$$

Adding, we have

$$a_1^2 + a_2^2 + a_3^2 - 2S_2 < 0.$$

Using

$$a_1^2 + a_2^2 + a_3^2 = S_1^2 - 2S_2,$$

we obtain,

$$6S_2 - S_1^2 > 4S_2 - S_1^2 > 0.$$

To show that α lies within the triangle, we need only prove that the real root is positive and less than the shortest side of the triangle.

Let

$$f(x) = 2x^3 - S_1x^2 + S_2x - S_3 = x^3 + (x - a_1)(x - a_2)(x - a_3),$$

and let a_k be the minimum of a_1, a_2, a_3. Since $f(x)<0$ when $x=0$, and $f(x)>0$ when $x=a_k$, it is evident that α lies within the triangle.

If we let

$$A_1\alpha_2' = A_2\alpha_3' = A_3\alpha_1' = y,$$

the resultant equation,

$$2y^3 - S_1y^2 + S_2y - S_3 = 0,$$

makes it obvious that α' also lies within the triangle, and the theorem is established.

THEOREM 2. *The points α and α' are isotomic conjugates.*

Proof. The two equations,

$$2x^3 - S_1x^2 + S_2x - S_3 = 0,$$
$$2y^3 - S_1y^2 + S_2y - S_3 = 0,$$

are identical, so that $x=y$, and α and α' are isotomic conjugates.†

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* L. E. Dickson, Elementary Theory of Equations, p. 34.

† A simple rational case arises for the triangle with sides 9, 14, 15, for which $x=y=6$.

THEOREM 3. *The locus of a point P such that $A_1P_3=A_2P_1$ is a conic, where P_i is the intersection of A_iP and A_jA_k, $i\neq j\neq k\neq i$.*

Proof. Using trilinear coördinates, with triangle $A_1A_2A_3$ as the triangle of reference, and with the center of the inscribed circle as the unit point, we see that the line A_1P passes through the points $A_1(1, 0, 0)$ and $P_1(0, [a_1-c]a_3, ca_2)$, and the equation of A_1P is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 0 \\ 0 & (a_1-c)a_3 & ca_2 \end{vmatrix} = 0, \quad \text{where} \quad A_1P_3 = A_2P_1 = c,$$

or

$$a_2x_2c + a_3x_3c - a_1a_3x_3 = 0.$$

Similarly, the equation of A_3P is found to be

$$a_1x_1c + a_2x_2c - a_3a_2x_2 = 0.$$

Eliminating c, we find that the locus of the point P is the conic

$$a_1x_3(a_1x_1 + a_2x_2) = a_2x_2(a_2x_2 + a_3x_3).$$

Similarly, the locus of a point Q, such that $A_2Q_1=A_3Q_2$, where Q_i is the intersection of A_iQ and A_jA_k, $i\neq j$, $j\neq k$, $k\neq i$, is found to be the conic

$$a_2x_1(a_2x_2 + a_3x_3) = a_3x_3(a_3x_3 + a_1x_1);$$

also, the locus of a point R, determined similarly, where $A_3R_2=A_1R_3$, is the conic

$$a_3x_2(a_3x_3 + a_1x_1) = a_1x_1(a_1x_1 + a_2x_2).$$

THEOREM 4. *The conic which is the locus of P passes through A_3 and is tangent to the line $x_3=0$ at A_1.*

This is evident by direct substitution in the equation for the locus of P.

Similarly, the locus of Q passes through A_1 and is tangent to $x_1=0$ at A_2; also, the locus of R passes through A_2 and is tangent to $x_2=0$ at A_3.

THEOREM 5. *The three conics which are the loci of P, Q, and R, are concurrent at α.*

Since, by definition,

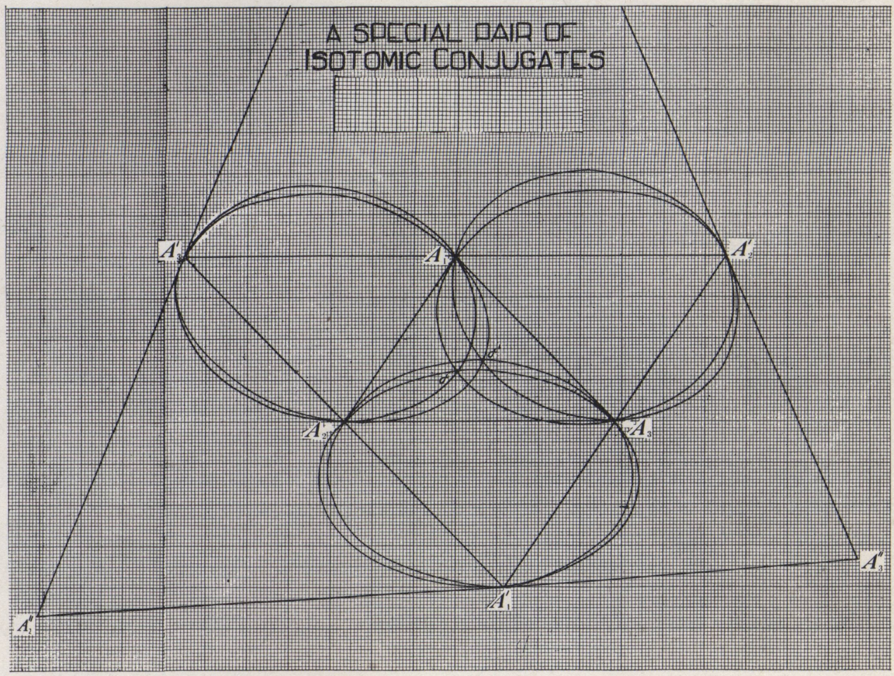
$$A_1\alpha_3 = A_2\alpha_1 = A_3\alpha_2,$$

α must be common to the loci of P, Q, and R.

If we consider the loci of P', Q', and R', such that

$$A_2P_3' = A_3P_1', \qquad A_3Q_1' = A_1Q_2', \qquad A_1R_2' = A_2R_3',$$

respectively, then we arrive at three more conics which will have the point α' in common. The loci of P', Q', and R' pass through the points A_1, A_2, and A_3, respectively, and are tangent to the lines $x_1=0$ at A_3, $x_2=0$ at A_1, and $x_3=0$ at A_2, respectively.



THEOREM 6. *At least two of the three conics which are the loci of P, Q, and R, (P', Q', and R') are ellipses.*

To determine what type of conic the locus of P is, we may solve simultaneously its equation,

$$a_1x_3(a_1x_1 + a_2x_2) = a_2x_2(a_2x_2 + a_3x_3),$$

with the equation of the line at infinity,

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

We see that* if $a_3 < 4a_1$, the locus of P is an ellipse;
if $a_3 = 4a_1$, the locus of P is a parabola;
if $a_3 > 4a_1$, the locus of P is a hyperbola.

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* S. L. Loney, Elements of Coördinate Geometry, Part II, p. 64.

In like manner, if $a_1 < 4a_2$, the locus of Q is an ellipse, and if $a_2 < 4a_3$, the locus of R is an ellipse.

It is evident that at most one of the three relations, $a_1 \geqq 4a_2$, $a_2 \geqq 4a_3$, $a_3 \geqq 4a_1$, can hold.

Similarly, at least two of the three conics which are the loci of P', Q', and R' are ellipses.

THEOREM 7. *The loci of P, Q, and R are tangent to the loci of P', Q', and R', respectively, at the points* $A_2'(1/a_1, -1/a_2, 1/a_3)$, $A_3'(1/a_1, 1/a_2, -1/a_3)$, *and* $A_1'(-1/a_1, 1/a_2, 1/a_3)$, *respectively.*

The locus of P,

$$a_1x_3(a_1x_1 + a_2x_2) = a_2x_2(a_2x_2 + a_3x_3),$$

and the locus of P',

$$a_3x_1(a_3x_3 + a_2x_2) = a_2x_2(a_2x_2 + a_1x_1),$$

both pass through the points A_1 and A_3. It can be shown that both conics are tangent to the line

$$a_1^2 x_1 + a_2(a_1 + a_3)x_2 + a_3^2 x_3 = 0, \quad \text{at} \quad A_2'(1/a_1, -1/a_2, 1/a_3).$$

Similarly, the tangent to the loci of Q and Q' at $A_3'(1/a_1, 1/a_2, -1/a_3)$ is the line

$$a_2^2 x_2 + a_3(a_2 + a_1)x_3 + a_1^2 x_1 = 0;$$

also, the tangent to the loci of R and R' at $A_1'(-1/a_1, 1/a_2, 1/a_3)$ is the line

$$a_3^2 x_3 + a_1(a_3 + a_2)x_1 + a_2^2 x_2 = 0.$$

The triangle formed by these three tangents is found to have the vertices $A_1''(-s/a_1, [s-a_3]/a_2, [s-a_2]/a_3)$, $A_2''([s-a_3]/a_1, -s/a_2, [s-a_1]/a_3)$, $A_3''([s-a_2]/a_1, [s-a_1]/a_2, -s/a_3)$, where $s = (a_1+a_2+a_3)/2$.

3. Horn angles. It is evident that the loci of P and P', Q and Q', R and R', form horn angles at A_2', A_3', A_1', respectively. Using the ratios of the curvatures at the points of tangency as the measures of these horn angles, we find that these measures, which are projective invariants, are

$$K_P/K_{P'} = a_1/a_3, \qquad K_Q/K_{Q'} = a_2/a_1, \qquad K_R/K_{R'} = a_3/a_2.$$

It follows that

$$K_PK_QK_R = K_{P'}K_{Q'}K_{R'}.$$

4. Perspective relations. Further investigation discloses that α and α' are instrumental in producing several interesting perspectivities. It is found that triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ are both in perspective with triangle $A_1''A_2''A_3''$ and with each other.

THEOREM 8. *Triangle* $A_1A_2A_3$ *is in perspective with triangle* $A_1''A_2''A_3''$ *with* G, *the Gergonne* point, as the center of perspective.*

If I_1 is the point of tangency of the inscribed circle and A_2A_3, then

$$\frac{\Delta A_1A_2I_1}{\Delta A_1A_3I_1} = \frac{s - a_2}{s - a_3} = \frac{\Delta GA_2I_1}{\Delta GA_3I_1} = \frac{\Delta A_1A_2I_1 - \Delta GA_2I_1}{\Delta A_1A_3I_1 - \Delta GA_3I_1} = \frac{\Delta A_1A_2G}{\Delta A_1A_3G} = \frac{a_3x_3}{a_2x_2} = \frac{s - a_2}{s - a_3},$$

where G, the Gergonne point, has the coördinates (x_1, x_2, x_3). Hence

$$x_2 : x_3 = \frac{1}{a_2(s - a_2)} : \frac{1}{a_3(s - a_3)};$$

and similarly,

$$x_1 : x_2 : x_3 : = \frac{1}{a_1(s - a_1)} : \frac{1}{a_2(s - a_2)} : \frac{1}{a_3(s - a_3)},$$

so that the Gergonne point may be represented as $G(1/a_1[s-a_1],\ 1/a_2[s-a_2],\ 1/a_3[s-a_3])$.

To show that A_1'', A_1, and G are collinear we need only prove that

$$\begin{vmatrix} 1 & 0 & 0 \\ -s/a_1 & (s - a_3)/a_2 & (s - a_2)/a_3 \\ 1/a_1(s - a_1) & 1/a_2(s - a_2) & 1/a_3(s - a_3) \end{vmatrix} = 0.$$

In like manner it can be shown that G lies on $A_2''A_2$ and $A_3''A_3$ and is therefore the center of perspective of triangles $A_1''A_2''A_3''$ and $A_1A_2A_3$. The axis of perspective is the line

$$a_1^2 x_1 + a_2^2 x_2 + a_3^2 x_3 = 0.$$

THEOREM 9. *Triangle* $A_1'A_2'A_3'$ *is in perspective with triangle* $A_1''A_2''A_3''$ *with* N, *the Nagel† point, as the center of perspective.*

The proof is similar to the proof of Theorem 8, with

$$\frac{\Delta A_1A_2N_1}{\Delta A_1A_3N_1} = \frac{s - a_3}{s - a_2} = \frac{a_3x_3}{a_2x_2},$$

where N_1 is the point of contact of the escribed circle on A_2A_3, and the coördinates of N are found to be $x_i = (s - a_i)/a_i$, $(i = 1, 2, 3)$.

The axis of perspective is the line

$$a_1^2(a_2 + a_3)x_1 + a_2^2(a_3 + a_1)x_2 + a_3^2(a_1 + a_2)x_3 = 0.$$

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* The Gergonne point is defined as the intersection of the lines joining each vertex of a triangle to the point of contact of the inscribed circle on the opposite side.

† The Nagel point is defined as the intersection of the lines joining each vertex of a triangle to the point of contact of the respective escribed circle on the opposite side.

THEOREM 10. *Triangle $A_1A_2A_3$ is in perspective with triangle $A_1'A_2'A_3'$ with M, the Median point, as the center of perspective.*

The proof is similar to the proof of Theorem 8, where the coördinates of M are found to be $x_i = 1/a_i$, $(i=1, 2, 3)$.

The axis of perspective is the line at infinity

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

5. **Constructibility.** Since A_1, A_2, A_3, are given and A_1', A_2', A_3', M, N, G, can be constructed with ruler and compass, it follows that we can find A_1'', A_2'', A_3'', with ruler and compass, for A_iG and $A_i'N$ meet at A_i'', $(i=1, 2, 3)$. The conics are known and easily constructible pointwise.

GENERALIZATIONS ON PARTIAL FRACTIONS IN DETERMINANTAL FORMS

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1. **Arbitrary quadratic factors.** Professor Turnbull has shown* how partial fractions can be expressed in determinantal form for functions of the form $f(x)/Q(x)$ where $Q(x)$ consists of simple (non-repeated or repeated) factors $(x-a_r)$ and $f(x)$ is a polynomial of order lower than that of $Q(x)$. He suggested that the method could be extended to functions $Q(x)$ which included complex factors, and for arbitrary pairs of complex roots a straightforward generalization seems to be the simplest process. But if there are purely imaginary roots, that is, if $Q(x)$ has factors of the form $(x^2+a_r^2)$ a modified process leads to simpler results. Such a modification is suggested here.

To recall Turnbull's method we shall first describe briefly, by means of an example, the method to be used in the case of arbitrary quadratic factors. For simplicity we shall assume that $Q(x)$ has the form

$$Q(x) = (x^2 + a_1x + b_1)(x^2 + a_2x + b_2),$$

and suppose that the roots of $x^2+a_1x+b_1=0$ and of $x^2+a_2x+b_2=0$ are s_1, t_1 and s_2, t_2 respectively. Then Turnbull's formula for the partial fractions may be written down immediately:

$$(1)\qquad \frac{f(x)}{Q(x)} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ s_1 & t_1 & s_2 & t_2 \\ s_1^2 & t_1^2 & s_2^2 & t_2^2 \\ \dfrac{f(s_1)}{x-s_1} & \dfrac{f(t_1)}{x-t_1} & \dfrac{f(s_2)}{x-s_2} & \dfrac{f(t_2)}{x-t_2} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 & 1 \\ s_1 & t_1 & s_2 & t_2 \\ s_1^2 & t_1^2 & s_2^2 & t_2^2 \\ s_1^3 & t_1^3 & s_2^3 & t_2^3 \end{vmatrix}.$$

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* H. W. Turnbull, Note on Partial Fractions and Determinants, Proceedings of the Edinburgh Mathematical Society, Series 2, vol. 1, pp. 49–54, May, 1927.

Elementary transformations of the determinants enable us to write the ratio in the form

$$(2)\qquad \frac{f(x)}{Q(x)} = \frac{1}{\Delta}\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & H_1^{(1)} & 1 & H_2^{(1)} \\ H_1^{(1)} & H_1^{(2)} & H_2^{(1)} & H_2^{(2)} \\ \dfrac{xF_1 - G_1}{x^2 + a_1x + b_1} & \dfrac{xK_1 - b_1F_1}{x^2 + a_1x + b_1} & \dfrac{xF_2 - G_2}{x^2 + a_2x + b_2} & \dfrac{xK_2 - b_2F_2}{x^2 + a_2x + b_2} \end{vmatrix},$$

where

$$F_k = \frac{f(s_k) - f(t_k)}{s_k - t_k}, \qquad G_k = \frac{t_kf(s_k) - s_kf(t_k)}{s_k - t_k}, \qquad K_k = \frac{s_kf(s_k) - t_kf(t_k)}{s_k - t_k},$$

and the quantities $H_k^{(r)}$ denote the elementary homogeneous expressions in s_k and t_k of the rth degree. Clearly these can be expressed in terms of the original coefficients of Q:

$$H_k^{(1)} = -a_k; \quad H_k^{(2)} = a_k^2 - b_k; \quad H_k^{(3)} = -a_k(a_k^2 - 2b_k); \text{ etc.}$$

The quantities F_k, G_k, K_k can also be expressed in terms of the original coefficients of f and Q.

The denominator Δ has the form

$$\Delta = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & H_1^{(1)} & 1 & H_2^{(1)} \\ H_1^{(1)} & H_1^{(2)} & H_2^{(1)} & H_2^{(2)} \\ H_1^{(2)} & H_1^{(3)} & H_2^{(2)} & H_2^{(3)} \end{vmatrix},$$

and its value is found to be

$$\Delta = (b_1 - b_2)^2 + (a_1 - a_2)(a_1b_2 - a_2b_1).$$

The form of the ratio f/Q in (2) can obviously be extended to the case of any number of quadratic factors in Q, and it can also be extended to cover the case of repeated factors.

2. Factors of the form x^2+a^2. When $Q(x)$ has purely imaginary roots, *i.e.*,

$$Q(x) = (x^2 + a_1^2)^{r_1}(x^2 + a_2^2)^{r_2} \cdots (x^2 + a_n^2)^{r_n},$$

we may write $x^2 = y$ and express $f(x)$ as

$$f(x) = g(x^2) + xh(x^2) = g(y) + xh(y).$$

To each part of f in this form the method for simple roots may be applied and the partial fraction formula obtained. This is not possible, however, if $Q(x)$ also includes simple factors such as $(x-c)$. In this case the following method may be adopted.

Assume that

$$Q(x) = (x^2 + a_1^2) \cdots (x^2 + a_m^2)(x - \alpha_{m+1}) \cdots (x - \alpha_n).$$

Then we may write

$$\frac{f(x)}{Q(x)} = \frac{(x + \alpha_{m+1}) \cdots (x + \alpha_n) f(x)}{\prod_1^n (x^2 - \alpha_r^2)},$$

where

$$\alpha_r^2 = -a_r^2, \qquad r = 1, 2, \cdots, m.$$

The numerator may now be expressed in the form $F(x^2)+xG(x^2)$, and therefore

$$\frac{f(x)}{Q(x)} = \frac{F(y)}{\prod_1^n (y - \alpha_r^2)} + x\frac{G(y)}{\prod_1^n (y - \alpha_r^2)}.$$

Each of the terms on the right can be expressed as partial fractions in determinantal form, and combining the two parts thus obtained we get the formula

$$\frac{f(x)}{Q(x)} = \frac{1}{\Delta}\begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \alpha_1^2 & \cdots & \alpha_m^2 & \alpha_{m+1}^2 & \cdots & \alpha_n^2 \\ \alpha_1^4 & \cdots & \alpha_m^4 & \alpha_{m+1}^4 & \cdots & \alpha_n^4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{2(n-2)} & \cdots & \alpha_m^{2(n-2)} & \alpha_{m+1}^{2(n-2)} & \cdots & \alpha_n^{2(n-2)} \\ \dfrac{F(\alpha_1^2) + xG(\alpha_1^2)}{x^2 - \alpha_1^2} & \cdots & \dfrac{F(\alpha_m^2) + xG(\alpha_m^2)}{x^2 - \alpha_m^2} & \dfrac{G(\alpha_{m+1}^2)}{x - \alpha_{m+1}} & \cdots & \dfrac{G(\alpha_n^2)}{x - \alpha_n} \end{vmatrix},$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \alpha_1^4 & \alpha_2^4 & \cdots & \alpha_n^4 \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{2(n-1)} & \alpha_2^{2(n-1)} & \cdots & \alpha_n^{2(n-1)} \end{vmatrix} = \begin{array}{r} (\alpha_2^2 - \alpha_1^2)(\alpha_3^2 - \alpha_1^2) \cdots (\alpha_n^2 - \alpha_1^2) \\ (\alpha_3^2 - \alpha_2^2) \cdots (\alpha_n^2 - \alpha_2^2) \\ \cdots\cdots\cdots\cdots \\ (\alpha_n^2 - \alpha_{n-1}^2). \end{array}$$

In the determinant in the numerator the element of the last row corresponding to $(x^2-\alpha_r^2)$ when $r>m$ has been written as $G(\alpha_r^2)/(x-\alpha_r)$. For, when $r>m$, $F(x^2)+xG(x^2)$ contains $x+\alpha_r$ as a factor, so that $F(\alpha_r^2)=\alpha_r G(\alpha_r^2)$ and $f(\alpha_r^2)+xG(\alpha_r^2)=(x+\alpha_r)G(\alpha_r^2)$.

Expansion of the determinant will lead to the partial fraction formula

$$\frac{f(x)}{Q(x)} = \sum_{r=1}^{m} \frac{F(-a_r^2) + xG(-a_r^2)}{A_r(x^2 + a_r^2)} + \sum_{r=m+1}^{n} \frac{G(\alpha_r^2)}{A_r(x - \alpha_r)},$$

where

$$A_r = (\alpha_r^2 - \alpha_1^2)(\alpha_r^2 - \alpha_2^2) \cdots (\alpha_r^2 - \alpha_{r-1}^2)(\alpha_r^2 - \alpha_{r+1}^2) \cdots (\alpha_r^2 - \alpha_n^2).$$

3. **Repeated factors.**. The case of repeated quadratic or linear factors can be dealt with in the same way as in Turnbull's original paper. An example will illustrate the process. Let

$$Q(x) = (x^2 + a_1^2)^2(x^2 + a_2^2)\,(x - b_1)^2(x - b_2)(x - b_3),$$

and write

$$a_1^2 = -\alpha_1, \qquad a_2^2 = -\alpha_2, \qquad b_1^2 = \alpha_3, \qquad b_2^2 = \alpha_4, \qquad b_3^2 = \alpha_5.$$

Then

$$\frac{f(x)}{Q(x)} = \begin{vmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \alpha_1 & 1 & \alpha_2 & \alpha_3 & 1 & \alpha_4 & \alpha_5 \\ \alpha_1^2 & 2\alpha_1 & \alpha_2^2 & \alpha_3^2 & 2\alpha_3 & \alpha_4^2 & \alpha_5^2 \\ \alpha_1^3 & 3\alpha_1^2 & \alpha_2^3 & \alpha_3^3 & 3\alpha_3^2 & \alpha_4^3 & \alpha_5^3 \\ \alpha_1^4 & 4\alpha_1^3 & \alpha_2^4 & \alpha_3^4 & 4\alpha_3^3 & \alpha_4^4 & \alpha_5^4 \\ \alpha_1^5 & 5\alpha_1^4 & \alpha_2^5 & \alpha_3^5 & 5\alpha_3^4 & \alpha_4^5 & \alpha_5^5 \\ \dfrac{F+xG}{x^2-\alpha_1} & \left(\dfrac{F+xG}{x^2-\alpha_1}\right)' & \dfrac{F+xG}{x^2-\alpha_2} & \dfrac{G}{x-\sqrt{\alpha_3}} & \left(\dfrac{G}{x-\sqrt{\alpha_3}}\right)' & \dfrac{G}{x-\sqrt{\alpha_4}} & \dfrac{G}{x-\sqrt{\alpha_5}} \end{vmatrix}$$

$$\div \begin{vmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \alpha_1 & 1 & \alpha_2 & \alpha_3 & 1 & \alpha_4 & \alpha_5 \\ \alpha_1^2 & 2\alpha_1 & \alpha_2^2 & \alpha_3^2 & 2\alpha_3 & \alpha_4^2 & \alpha_5^2 \\ \alpha_1^3 & 3\alpha_1^2 & \alpha_2^3 & \alpha_3^3 & 3\alpha_3^2 & \alpha_4^3 & \alpha_5^3 \\ \alpha_1^4 & 4\alpha_1^3 & \alpha_2^4 & \alpha_3^4 & 4\alpha_3^3 & \alpha_4^4 & \alpha_5^4 \\ \alpha_1^5 & 5\alpha_1^4 & \alpha_2^5 & \alpha_3^5 & 5\alpha_3^4 & \alpha_4^5 & \alpha_5^5 \\ \alpha_1^6 & 6\alpha_1^5 & \alpha_2^6 & \alpha_3^6 & 6\alpha_3^5 & \alpha_4^6 & \alpha_5^6 \end{vmatrix},$$

where in the last row of the numerator the argument of F and G is always the particular α which appears in the denominator, and a prime denotes differentiation with respect to that α. The value of the determinant in the denominator is

$$\Delta = (\alpha_2 - \alpha_1)^2(\alpha_3 - \alpha_1)^4(\alpha_4 - \alpha_1)^2(\alpha_5 - \alpha_1)^2(\alpha_3 - \alpha_2)^2(\alpha_4 - \alpha_2)(\alpha_5 - \alpha_2) \times (\alpha_4 - \alpha_3)^2(\alpha_5 - \alpha_3)^2(\alpha_5 - \alpha_4).$$

To find the coefficient of $(x^2+a_1^2)^{-2}$ we proceed as follows. By definition

$$\left(\frac{F(\alpha_1) + xG(\alpha_1)}{x^2 - \alpha_1}\right)' = \frac{F'(\alpha_1) + xG'(\alpha_1)}{x^2 - \alpha_1} + \frac{F(\alpha_1) + xG(\alpha_1)}{(x^2 - \alpha_1)^2}.$$

The cofactor of this term in the determinant in the numerator is

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & 1 & \alpha_4 & \alpha_5 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & 2\alpha_3 & \alpha_4^2 & \alpha_5^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & 3\alpha_3^2 & \alpha_4^3 & \alpha_5^3 \\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & 4\alpha_3^3 & \alpha_4^4 & \alpha_5^4 \\ \alpha_1^5 & \alpha_2^5 & \alpha_3^5 & 5\alpha_3^4 & \alpha_4^5 & \alpha_5^5 \end{vmatrix},$$

which has the value

$$(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)^2(\alpha_4 - \alpha_1)(\alpha_5 - \alpha_1)(\alpha_3 - \alpha_2)^2(\alpha_4 - \alpha_2)(\alpha_5 - \alpha_2)(\alpha_4 - \alpha_3)^2 \times (\alpha_5 - \alpha_3)^2(\alpha_5 - \alpha_4).$$

Thus the coefficient of $(x^2+a_1^2)^{-2}$ is

$$\frac{F(\alpha_1) + xG(\alpha_1)}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)^2(\alpha_4 - \alpha_1)(\alpha_5 - \alpha_1)}.$$

Evidently this method can be generalized to any combination of repeated factors.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

THE DEADLY PARALLEL

A. J. KEMPNER, University of Colorado

Readers of the MONTHLY may be interested in two excerpts from the daily press:

I. (From the *Rocky Mountain News*, Denver, Friday, February 10, 1939). At the Annual Regional Study Conference of the Progressive Education Association, Denver, Colorado, Dr. Lois Hayden Meek, Professor of Education at Columbia University, is quoted as saying: "Intelligent supervision and encouragement of such group relationship at that time is of far greater importance in the future education of the youth than cut and dried history or mathematics. . . . It is my opinion we should not have so many specialists on our teaching staffs. . . . A better method would be to have one teacher teach more subjects and have longer and more frequent contact with pupils, studying more carefully the individual problems of each. *The function of such a teacher would not hinge so much on knowing the subjects to be taught as on knowing how to arrange and present such subjects so they could be most readily learned.*" (Italics mine.)

II. (From the *Denver Post*, Saturday, February 11, 1939, reporting opinions of students at the same conference).

A high school senior from Northern Colorado: "Everything would be fine if we had good teachers. You've been talking a lot about dull subjects that should be dropped from the curriculum. Those subjects are dull because the teachers are dull. If all our teachers were alert and interesting, there would be no such thing as wasted courses. Even a course in Latin would be fun and would accomplish real education if the teacher presented it the right way."

A college student: "It's all a matter of the personality of the teacher. All of these proposed drastic changes in the method of education would be unnecessary if teachers generally had a real understanding of what young people want. We don't resist learning. As a matter of fact we're anxious to learn. All we ask is that there be some pleasure, some intellectual excitement in the process. . . . Every student knows there are some teachers in whose classes he would like to spend much more time than allotted. When the student fidgets and waits impatiently for the bell to ring, it is the fault of the teacher, not the fault of the subject."

May it be that we have overlooked our best allies in our fight for a sound education based on fundamentals,—the students themselves?

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A CORRECTION BY PROFESSOR CANDY

A. L. CANDY, University of Nebraska

For the benefit of those readers of the MONTHLY who have copies of my book *Magic Squares of an Even Order*, I wish to make the following correction: The last four Basic Squares on p. 91, *viz.*, Nos. 5, 6, 7, 8, should be stricken out, because they are duplicates, respectively, of Nos. 2, 1, 4, 3, on the same page, since they give the same "Groups of Four" when rows and columns are permuted so as to retain the same principal diagonals. This reduces the total number E. D. S. (p. 95) from 896 to 880, which agrees with the number obtained by D. H. Lehmer and others. Hence, wherever this number 896 appears elsewhere in this book, it should be changed to 880. See especially pp. 37, 174, and 175.

I am very glad to have this opportunity of making this public acknowledgement of my error.

A CORRECTION BY PROFESSOR MUSSELMAN

J. R. MUSSELMAN, Western Reserve University

I wish to call attention to an error in my article, "On the line of images," this MONTHLY, vol. 45, pp. 421–430. On page 430, line 2 following should read: "of the Kiepert hyperbola and at the point whose line of images as to $A_1A_2A_3$ is the line joining the orthocenter and the symmedian point of $A_1A_2A_3$."

A THEOREM ON GRAPHS, WITH AN APPLICATION TO A PROBLEM OF TRAFFIC CONTROL

H. E. ROBBINS, Institute for Advanced Study

It is the object of this note to answer a question* which is suggested by the problem of traffic control in a modern city: "When may the arcs of a graph be so oriented that one may pass from any vertex to any other, traversing arcs in the positive sense only?" Any graph with this property will be called orientable. The answer is provided by the following theorem:

THEOREM. *A graph is orientable if and only if it remains connected after the removal of any arc.*

Let us suppose that week-day traffic in our city is not particularly heavy, so that all streets are two-way, but that we wish to be able to repair any one street at a time and still detour traffic around it so that any point in the city may be reached from any other point. On week-ends no repairing is done, so that all streets are available, but due to the heavy traffic (perhaps it is a college town with a noted football team) we wish to make all streets one-way and still be able to get from any point to any other without violating the law. Then the theorem states that if our street-system is suitable for week-day traffic it is also suitable for week-end traffic and conversely. We proceed to give a few definitions and a proof of the theorem.

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* Proposed by S. Ulam. For a simplification in the proof we are indebted to H. Whitney.

By a graph G we mean a (finite) set of objects $\langle p, q, \cdots \rangle$ called *vertices*, together with a (finite) set of objects called *arcs*, which join certain pairs of distinct vertices. We shall represent an arc joining p and q by (pq), with the understanding that there may be more than one such arc. G is *connected* if, given any two of its vertices p, q, there exists a chain of arcs joining p and q of the form

$$(pr_1)(r_1r_2) \cdots (r_sq). \tag{1}$$

G is *oriented* if a direction is assigned to each arc, symbolized by choosing one of the two representations (pq) and (qp) of the unoriented arc, to be called the *admissible* representation. G is *orientable* if it may be oriented in such a way that any two of its vertices may be joined by a chain (1) of arcs, each in its admissible representation. The definitions of *subgraph*, and of the special subgraph obtained from G by removing one of its arcs, are obvious.

Now referring to the theorem, we see that the necessity of the condition is clear, for if G is disconnected by the removal of an arc (pq), then no matter which representation of (pq) we call admissible in an orientation of G, passage either from p to q or from q to p by a chain of admissible arcs will be impossible, so that G is not orientable. It remains to prove the sufficiency of the condition. Suppose G remains connected after the removal of any arc. (Then *a fortiori* G is connected.) Choose a vertex p of G and consider the class of all subgraphs of G which include p and which are orientable. This class is non-void, since the subgraph consisting of p alone satisfies the conditions. Let G' be a sub-graph in this class with maximal number of vertices. We may assume that G' contains all arcs of G whose vertices are in G', since otherwise any such arcs which did not belong to G' could be added, oriented arbitrarily, and G' would remain orientable. Then G' must be identical with G. For suppose p^* is a vertex of G not in G'. Join p^* to p by a chain

$$(p_1p_2)(p_2p_3) \cdots (p_{n-1}p_n), \qquad [p_1 = p^*; p_n = p], \tag{2}$$

and order the arcs of (2) from left to right. Let (p_kp_{k+1}) be the last arc of (2) which is not in G'. Then p_k does not belong to G', while p_{k+1} does. By hypothesis, there exists a chain of arcs

$$(\bar{p}_1\bar{p}_2)(\bar{p}_2\bar{p}_3) \cdots (\bar{p}_{m-1}\bar{p}_m), \qquad [\bar{p}_1 = p_k; \bar{p}_m = p_{k+1}], \tag{3}$$

which does not include (p_kp_{k+1}). Let $(\bar{p}_s\bar{p}_{s+1})$ be the last arc of (3) which does not belong to G'. Then the subgraph G^* consisting of G' plus $(p_{k+1}p_k)$ and the chain

$$(\bar{p}_1\bar{p}_2)(\bar{p}_2\bar{p}_3) \cdots (\bar{p}_s\bar{p}_{s+1}), \tag{4}$$

where these arcs are to be oriented as written, is clearly orientable, contains p, and has more vertices than G', which is a contradiction and completes the proof.

EVEN PERMUTATIONS AS PRODUCTS OF CYCLES

Leonard Miller, Brooklyn, New York

Theorem. *Every even permutation P on n distinct letters can be written as the product of an even number of cycles on these n letters, each cycle being of order m, where* $m=2, 3, \cdots$, *or* n.

Proof. By definition, any even permutation can be written as the product of an even number of transpositions. It is also known that any even permutaton can be written as the product of cycles of three letters each. The theorem is therefore true for $m=2, 3$. We will show that any even permutation written as the product of an even number of cycles of r letters each, where $2 \leqq r < n$, can be written as the product of the same number of cycles of $r+1$ letters each. The theorem will then follow by induction. Since the permutation group is of degree n, the theorem loses significance for $r \geqq n$.

Suppose that the permutation is written as the product of an even number of cycles each of order r. Let us group the cycles into pairs. Any given pair of cycles will fall into one of the following two types.

I. Pairs of cycles in which all of the letters of the first cycle are repeated in some order in the second.

II. Pairs of cycles in which at least one letter of the first cycle is not repeated in the second.

We will now show that every pair of cycles of type I and type II can be written as a pair of cycles of order $r+1$.

Take any pair of cycles of type I,

$$(a_1 a_2 \cdots a_r)(a_1' a_2' \cdots a_r'). \tag{1}$$

Since the letters of the first cycle are the same as the letters in the second, we may suppose that $a_1 = a_1'$. Since $r < n$, there is an a_{r+1}. If we form

$$(a_1 a_2 \cdots a_r a_{r+1})(a_1 a_{r+1} a_2' \cdots a_r'), \tag{2}$$

it is easy to verify that (1) = (2).

Take any pair of cycles of type II,

$$(a_1 a_2 \cdots a_i \cdots a_r)(a_1' a_2' \cdots a_i' \cdots a_r'). \tag{3}$$

Then let a_i be the last letter in the first cycle which is not found in the second, and let a_j' be the last letter in the second which is not found in the first. If we form

$$(a_1 a_2 \cdots a_j' a_i \cdots a_r)(a_1' a_2' \cdots a_j' a_i \cdots a_r'), \tag{4}$$

it is easy to verify that (3) = (4), and so the theorem follows.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Mathematical Adventures. By Fletcher Durell, Boston, Bruce Humphries, 1939, 157 pages. $2.00.

A Course in General Mathematics. By Clinton Harvey Currier, Emery Ernest Watson, and James Sutherland Frama. Revised Edition. New York, Macmillan, 1939. 10+382 pages. $3.00.

Your Chance to Win: The Laws of Chance and Probability. By Horace C. Levinson. New York and Toronto, Farrar and Rinehart, 1938. 343 pages. $2.50.

Lezioni di Analisi Matematica, Parte Prima. By Franscesco Tricomi. Fourth Edition. Padova, Cedam, 1939. 8+328 pages. Lire 70.

REVIEWS

Formelsammlung zur praktischen Mathematik. By Günther Schulz. (Sammlung Göschen, 1110.) Berlin, de Gruyter, 1937. 147 pp.

This little book of the Göschen series gives the principal methods and formulas of numerical mathematics. Its seven chapters cover aids in computation, adjustment of observations, interpolation, solution of numerical equations, numerical integration and summation, representation of functions by approximating formulas, and the numerical solution of differential equations. The inherent errors or remainder terms of the processes and formulas are given in many instances.

Illustrative examples are worked out under some of the more important topics. References to some of the literature (mostly German) are given at frequent intervals, and a longer list of references (some of little value) are given at the end of the book.

A valuable feature of the book is the section on the numerical solution of partial differential equations (12 pages). The equations treated are second-order equations of the elliptic, hyperbolic, and parabolic types.

J. B. SCARBOROUGH

Medieval Number Symbolism. Its Sources, Meaning and Influence on Thought and Expressions. By Vincent Foster Hopper. New York, Columbia University Press, 1938. 12+241 pp.

The author states in the "Preface" that "it is the purpose of this study to reveal how deeply rooted in medieval thought was the consciousness of numbers, not as mathematical tools, nor yet as the counters in a game, but as fundamental realities, alive with memories and eloquent with meaning."

The author presents at the outset the origins of medieval number philosophy in three phases which are designated "elementary," astrological, and Pythagorean. The first of these is concerned with certain groups of objects which we all possess such as 5 fingers, easily identified with "hand"; the second goes back to the science of astrology among the Babylonians, a science which involves numbers associated with the heavenly bodies, among which appears the 7, the number of the then known planets; the third "fixed the relationship of the numbers to one another," as in particular, three numbers which are the sides of a right triangle.

These introductory chapters are followed by penetrating studies of "The Gnostics," "The Early Christian Writers," "Medieval Number Philosophy." As these topics indicate, the work is wholly a study in the belief held for centuries that number is the first principle of all things. The illustration from Ramus is well known—that God perfected the world in 6 days because 6 is a perfect number, a case in which mathematical reasoning gave added justification for the number. "Medieval philosophers and theologians understood the purely mathematical aspects of number to be of divine origin." An outstanding development of this belief is found in the science (not exact) of gematria to which present day numerology is a distant kin.

The mark of this belief in the power of number is found to a certain degree in the sciences and pseudosciences of the Middle Ages although its chief outlet was theology. The purely scientific attitude today repudiates any such belief.

The work culminates in a chapter entitled "The Beauty of Order—Dante" and uses the phrase "Dante's sharp and studied usage of significant numbers." Here a commentary is offered by the author as a suggestive method rather than a final statement of at least one "gate and key" to the Dantesque vision of the universe. The reviewer is no judge of the value of these interpretations. To a specialist in the field of Dante's works, they hold fascinating possibilities.

An appendix is devoted to "Number Symbols of Northern Paganism."

The book is of interest to anyone in the mathematical as well as any other field to whom any study on concepts in his field widens his horizon of thought and knowledge.

LAO G. SIMONS

Leitfaden der Stereometrie. By W. Benz. Zürich and Leipzig, Orell Füssli. 1938. 215 pages, 122 figures. Fr. 3.80.

The present text is designed to follow a similar one for plane geometry written by Gonseth and Marti, published by the same firm. The elements of plane analytic geometry, harmonic properties, and plane affinity are presupposed known; sines and cosines are used.

The approach to the subject is very different from that familiar to American students. No exercises are given, and little emphasis is laid on mensuration. On the other hand, great stress is laid on construction; sufficient details are provided to enable the pupil to make any construction in the book.

The first chapter begins with the principles of skew perspective, without axioms, and with only a few definitions; intuition is freely drawn upon. This is followed by the concept of a plane, defined as the locus of points determined by a linear range of points and one other point. After having established the usual descriptive properties, perpendicularity is introduced. Parallels are first brought in on page 50. Motion plays an important part. Infinite elements are first introduced intuitively, but are firmly established later in the chapter.

The second chapter considers problems in drawing and plane representation—practically identical with the ordinary h, v projections of descriptive geometry, applied to rectilinear figures. The third chapter applies these principles to bodies and surfaces of revolution; the fourth is similarly applied to solid angles. Prisms and cylinders, pyramids and cones are now treated rapidly, but when the little book is completed, the reader will have met all the properties ordinarily discussed in our texts, and also acquired a comprehensive knowledge of drawing and of mapping.

The text is clear, the style simple and pleasing; the proof-reading has been well done, and the figures are excellent. American text-book writers of solid geometry could profit from a complete mastery of this book.

VIRGIL SNYDER

College Algebra. By Louis J. Rouse. New York, John Wiley and Sons; London, Chapman and Hall. Second Edition, 1939. 13+462 pages. $2.25.

The first edition of this book appeared in 1931; it was reviewed in this MONTHLY, vol. 39, 1932, pp. 23–24, by Mildred Waters Dean. In 1934 a reprinting appeared, with a chapter on logarithms added.

The present edition retains the essential features of the earlier one, but the exercises are new, and several chapters have been added, including one on compound interest and annuities, one on probability and one on partial fractions. The chapter on determinants has been completely re-written, based on induction rather than inversions. The printing is excellent, and the proof-reading has been well done.

VIRGIL SNYDER

Advanced Algebra. Volume I by C. V. Durell. 8+193+22 pages. 4s. 6d. Volumes II and III by C. V. Durell and A. Robson. London, G. Bell and Sons, Ltd. 1938. 11+315+27 pages. 12s. 6d.

Volume I was originally published in 1932. The material covered corresponds roughly to that found in the last half of the conventional college algebra, but the treatment is more difficult. For example, calculus methods are used extensively in Chapter VI on "Logarithmic and Exponential Functions" and in Chapter VII on "Rational Functions."

Some idea of the scope of Volumes II and III may be obtained from the following chapter titles, of which the first five are from Volume II: Finite Series, Difference Equations, Factors and Partial Fractions, Theory of Equations, Sequences, Inequalities, Determinants, Matrices, Choice and Chance, Theory of

Numbers. A wealth of material is here presented for advanced undergraduates and graduates.

Developing theory by means of brief but well chosen paragraphs, the authors include more topics in each chapter than might be expected from the actual number of pages devoted to exposition.

As one has learned to expect from English textbooks, the emphasis is on exercises. Volume I contains 35 sets of Test Papers. Volumes II and III contain about 1500 exercises graded into three sections. The authors include more than the usual number of illustrative examples. Answers are given to all exercises, and books of hints are available for the last two volumes.

It is not likely that many teachers would be willing to give up the traditional courses in theory of equations and higher algebra. However, for supplementary material, reference, and as a source book for exercises, the authors have added a three volume work of unquestionable merit to their list of successful textbooks.

F. A. LEWIS

Magische Quadrate und Magische Parkette. By Gerhard Kowalewski. Leipzig, K. F. Koehlers Antiquarium, 1937. 78 pp. Rm. 2.

This is the second number in the series *Scientia Delectans* which has for its purpose the presentation of popular essays on various topics which in English we usually class under the term, Mathematical Recreations.

The present little book is devoted to the study of certain classes of magic squares of order four. These classes arise naturally from what the author calls the "cross condition" (Kreuzbedingung). Let the elements of the first row of the magic square be denoted by a_1, a_2, a_3, a_4, and the elements of the second, third and fourth rows in a similar way with the letters, b, c and d. Then if $a_2+d_3=b_4+c_1=s$, the square is said to satisfy the cross condition. If this holds, it is found that the relation $a_3+d_2=b_1+c_4=s'$ is also satisfied. Now if the difference $s-s'$ is zero, the square is said to be a Dürer square, since it is of the type which appears in Albrecht Dürer's engraving "Melancholy." All squares of this type are determined, and are shown to be identical with what are here called antisymmetric squares, but which have long been known in the literature as symmetric.

If the above difference $s-s'$, instead of having its least value zero, has its greatest possible value we have a new class of squares which the author names Sun squares. All squares of this type are determined and are found to be sixteen in number.

Certain, but not all, of the Sun squares are pandiagonal. This leads the author to a discussion of pandiagonal squares and the so-called magic parquetry or mosaic. The most interesting part of the book in connection with the pandiagonal squares, however, is their representation by means of a four-dimensional cube. Such a cube has sixteen vertices, and a fourth order magic square has sixteen elements. It is shown how these elements may be distributed over the sixteen corners of the four dimensional cube in such a way that the elements at the corners of each face of the bounding three dimensional cubes give the magic sum. These face squares may be arranged in three systems, or nets, from each one of which may be written down a fundamental pandiagonal magic square. Instead of the four dimensional cube we may deal with its three dimensional representation, namely, a cube with a square hole through it. This three dimensional figure may then be used to write down, by a very simple rule, all pandiagonal magic squares of order four.

The book is extremely well written. The explanations are very clear and presuppose absolutely no previous knowledge of the subject. Besides the main topics noted above, there are many details which we feel sure will delight anyone interested in magic squares.

The reviewer feels that it would have been worth-while, on page 31, to have pointed out that the relation satisfied by the four central elements and the four corner elements of a Dürer square holds for all magic squares of order four. A similar remark applies to the statement on page 68 concerning the sum of the four elements at the ends of the two middle columns (or rows). In regard to the term "Sun square" it seems that its choice may have been ill-advised, since Cornelius Agrippa, in 1531, in his *De Occulta Philsophia* associated a square of the *sixth* order with the sun.

G. E. Raynor

Der Keplersche Körper und Andere Bauspiele. By Gerhard Kowalewski. Leipzig, K. F. Koehlers Antiquarium, 1938. 65 pp. Rm. 2.

This is the third number in the series, *Scientia Delectans.*

Professor Kowalewski has succeeded in packing into the space of 65 pages an amazing amount of interesting and instructive material. The central and unifying theme of the book is the historically important Kepler solid. This is a convex solid with thirty congruent rhombic faces, the shorter diagonals of the rhombs forming the edges of a regular dodecahedron, and the longer diagonals the edges of a regular icosahedron. The ratio of these two diagonals is that of the well-known Golden Section, namely $\frac{1}{2}(\sqrt{5}-1)$ to 1.

It is shown how the Kepler solid may be built up from two sets, of ten each, of vari-colored parallelopipedal blocks, in such a way that two neighboring blocks touch along like colored faces. Other problems arise by distributing certain sets of markers over the thirty faces of the Kepler surface. For this purpose plane representation of the surface is used, one vertex being the point at infinity. Some of the distribution problems described are by no means simple, and are not likely to be solved by anyone not familiar with the theory explained by the author. The connection of the Kepler solid with a six-dimensional cube is another interesting high-light of the book.

It is impossible in a short review of this sort to give an adequate idea of the many interesting details. The reviewer thoroughly enjoyed reading the book and heartily recommends it to anyone interested in things mathematical.

G. E. Raynor

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N.J.

NOTICE TO ALL CLUBS. The spring semi-annual letter to all mathematics clubs should have reached you by the middle of May. This department would appreciate it if the secretary of each organization would see to it that the year's report is mailed to this office before he leaves his campus if it is possible. If there are any clubs who failed to receive this letter through some error, will they please communicate with this department as soon as possible, so that the yearly report will not be too long delayed.

CLUB TOPICS

42. SYMBOLIC LOGIC

SAUNDERS MAC LANE, Harvard University

Symbolic logic, born to be a universal language and a sort of calculus of all reason, bred in a turbulent atmosphere with controversies and paradoxes galore, now approaching maturity with the attainment of subtle methods and profound results, is a subject which has many attractions for mathematician and philosopher, for student, specialist, and layman. The logical analysis of demonstrations and of mathematical reasoning has an immediate interest to anyone who has dealt with mathematical proofs. Current research in logic is necessarily precise and technical, and often involves a heavy symbolic formalism, but the fundamental questions considered are generally direct and simple, and can be well understood by the uninitiate. Consider, for instance, Gödel's results which state (roughly speaking) that some theorems are true but not provable by any standard methods, and the intuitionists' contention that some proven theorems are not true! Again, raise the question whether there are mathematical problems which are insoluble. Questions like these are sure to provoke good discussion.

To assist mathematics clubs in discussing logic we give here a list of possible topics and a bibliography. The topics vary in difficulty and are more or less independent of each other. The bibliography is intended to give a fair representation of readable books and articles useful as an introduction to symbolic logic; we hope that it may also serve those who wish to study the subject independently. For complete bibliography we refer to Church [**15**].*

1. Paradoxes. A presentation of some of the standard paradoxes offers great opportunities. What happens to the beard of the barber who obligingly shaves all the men in town who do not shave themselves? This paradox and many others can be readily formulated in elementary terms; they are sure to produce discussion in a meeting. An excellent and readable discussion of the various standard paradoxes may be found in Fraenkel [**24**, §13]. In chapter 2 of the introduction of *Principia Mathematica* [**59**] there is (on pp. 60–61) a compact statement of a number of paradoxes, which may be read without a knowledge of the rest of this introduction. The same paradoxes are also stated in section 1 of [**49**]. Paradoxes are also stated by Menger [**44**], Bell [**2**, pp. 575–576], Black [**9**, pp. 97–101], Poincaré [**46**, *Science and Method*, chapter 5, parts 5 and 10]. Among the more sophisticated paradoxes is that due to Richard, which indicates that any system of symbolic logic is in some sense inadequate (Church [**14**]).

The classical solution of the paradoxes depends on the theory of types. To avoid paradoxical statements which refer to themselves in a circular fashion this theory separates the objects of discourse into various levels. In general outline, this idea can be presented as a natural one.* The original form of the type theory involve a difficult axiom of reducibility.† Another reasonable method of resolving the paradoxes appears in Behmann [**1**].

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* Numbers enclosed in brackets refer to the corresponding items of the bibliography below.

2. Calculus of propositions. An elementary and useful branch of logic is the calculus of propositions. This calculus offers methods of symbolizing the connectives "and," "or," "not," "if · · · then · · · ", and their various properties. Its methods can be readily explained to beginners. They are best found in some of the systematic treatments.‡ A modern treatment of this calculus in a form especially easy to use for mathematical proofs is given by Bernays's mimeographed lecture notes [**5**, pp. 7–47]. There are several very useful such sets of up-to-date lecture notes available.§ Brief treatments of the calculus of propositions may be found, for instance, in Menger [**44**] or in Black [**9**, pp. 41–48]. The newest text on symbolic logic is Bennett-Baylis [**3**].

3. Propositional functions. More fundamental than the calculus of propositions is that of propositional functions, which deals with statements involving a variable x and with "quantifications" of such statements by such phrases as "so-and-so holds for every x" or "there exists an x such that so and so." The elementary properties of these quantifiers are illuminating, especially in connection with the definition of limits, of uniform continuity, and of similar mathematical notions. An excellent and simple introduction is given by Tarski [**55**].‖ Other discussions of propositional functions are given by¶ Bernays [**5**, pp. 47–56] and Eaton [**22**, chapter 2].

4. Formalization of mathematics. An elementary but important aspect of symbolic logic is the possibility of using logical symbols to express in perfectly precise and completely symbolic form any mathematical theorem (or, for that matter, any proof). An interesting topic for club discussions would be the development of just enough logical symbolism to show how this could be done, with examples of mathematical proofs and definitions in precise form. The original ideas of Peano in this direction are discussed in chapter 1 of Lewis-Langford [**42**]. In Bernays [**5**, p. 2] is an excellent elementary logical treatment of groups, in Tarski [**55**] a similar discussion of real numbers. Carnap's Abriss [**11**] gives illustrations of the logical analysis of topology, point-set theory and projective geometry (§§31–35). More elementary examples of mathematical formalism appear in Hilbert-Bernays [**34**, §1], which gives careful and detailed explanations of the development involved. A classic American example of mathematics in logical form is E. H. Moore's *General Analysis*.* These symbolic techniques have even found use in the science of biology (Woodger, [**60**]).

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* Eaton [**22**, chapter 3, §4]; Menger [**44**]; Lewis-Langford [**42**, chapter 13].

† Principia Mathematica [**59**, chapter 3 of the introduction]; Jørgensen [**39**, volume 3, chapter 12], Black [**9**, pp. 97–104].

‡ Such as Hilbert-Ackermann [**33**, chapter 1]; Eaton [**22**, chapter 1]; Principia Mathematica [**59**, part 1, Section A (which can be read independently of the long introduction)]. A concise, efficient presentation of the Whitehead-Russell calculus is given in Carnap [**11**, §§1–5].

§ Rosser [**47**], Cooley [**17**]. The latter contains a large collection of verbal problems in symbolic logic.

‖ For those who can read German this is probably the best elementary introduction to symbolic logic.

¶ And also in most of the systematic treatises mentioned above: Lewis-Langford [**42**, chapter 5]; Lewis [**41**, chapter 4]. Cf. also Black [**9**, pp. 41–68].

5. Definitions of number. One of the earliest and most controversial applications of logic consists of an answer to the old problem, "What is number?" The logistic definition of number in terms of correspondence can be presented in a simple fashion, as in the first chapters of Russell [**50**]. These notions go back to Cantor, whose introduction of transfinite numbers by similar methods led to many violent controversies. The ideas behind these disputes are engagingly described by Bell [**2**, pp. 564–568]. Other approaches to the number concept are the Peano postulate method;† the historically significant method of Dedekind [**18**], and the radical approach of the intuitionists‡ (see below).

6. Logic of real numbers. Of fundamental importance for calculus and analysis is the application of logic to the definition of the real numbers. The most attractive, simple, and precise method today is that by means of postulates. Using elementary logic, Tarski presents in [**55**] two sets of postulates for the real numbers. Another approach§ uses the Dedekind cut idea to define numbers in terms of the logical properties of classes; it is brilliantly described in Russell [**50**, chapter 8]. This approach, however, involves all the difficulties of set-theory; a demonstration of its consistency (see below) is probably even more difficult than in the postulational approach.

In any club discussion, a good starting point would be the discussion of the paradoxes of Zeno.‖ The puzzling argument of the hare and the tortoise shows very clearly how necessary it is to think carefully and logically about the continuum of all points on the line.

The absolutely general notion of a "class" or "set" which appears in these investigations of numbers is the source of many difficulties in symbolic logic.¶ It offers a prime elementary example of mathematics in its most abstract and general form.

7. Boolean algebra. A different and mathematically stimulating approach to logic is that through Boolean algebra. It can be readily explained how "algebra" can be done with classes instead of numbers as objects. The laws of this algebra, which agree but in part with those of ordinary algebra, furnish a fine illustration of the possibility of new mathematical disciplines.** The foundations of Boolean algebra have been a favorite problem in postulate-theory. Innumerable systems of postulates for this algebra have been set up.* In [**38**], Huntington discusses the methods involved. For club talks the use of this Boolean algebra to solve explicitly complicated verbal problems is an attractive possibility.† Consider, for instance, its use in the insurance problems with which actuaries contend! (Berkeley [**4**]).

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* General Analysis, Part I, by Eliakim H. Moore; Memoirs of the American Philosophical Society, Philadelphia, 1935.

† Hilbert-Bernays [**34**, p. 219 ff.]; mathematical developments of the postulates may be found in E. Landau, Grundlagen der Analysis, Leipzig, 1930.

‡ Weyl [**57**, §6]; Poincaré [**46**, various sections on complete induction].

§ Still another procedure appears in Huntington [**35**].

‖ Cf. the profound treatment in Russell [**48**, chapters 33, 34, and 35].

¶ Discussions of classes in Russell [**50**, chapters 15 and 17]; Huntington [**35**, chapter 2], Russell [**48**, chapters 6 and 7], Weyl [**57**, §8].

** Treatments of Boolean algebra are given in Couturat [**16**]; Eaton [**22**, chapter 3]; Lewis-Langford [**42**, chapter 2]; Lewis [**41**, chapter 2]. The algebra in its historical setting appears in Bell [**2**, chapter 23]. Chapman-Henle [**13**, chapter 11], gives an elementary account.

Important recent developments have been made in Boolean algebra, by applying to it the powerful methods of modern abstract algebra (Stone [**53**]). Another startling extension lies in using more "truth values" than the ordinary "true" and "false." If ordinary Boolean algebra is considered as an algebra of statements (or propositions), every statement‡ is "equal" either to 0 (falsity) or to 1 (truth). It is possible to construct similar algebras using more than two such truth values.§ Hence the suggested topic: How can algebra treat statements which are neither true nor false (but are part way between)?

8. Intuitionism. A radical and startling attack on the methods of symbolic logic and of classical mathematics is offered by Brouwer and the intuitionists. They propound problems which are real posers. Consider the statement "somewhere in the decimal development of π appears a sequence of digits 1 2 3 4 5 6 7 8 9." Is it false?—Nobody could calculate π all the way to infinity just to disprove it. Is it true?—No one has yet found such a sequence. Perhaps it is neither true nor false! The writings of the intuitionist abound in many such neat dilemmas, sure to catch the eye of an inquisitive student.

The constructive ideas of the intuitionists are often pontifical and obscure, but some of their positions are clear.‖ Thus "*A* implies the absurdity of the absurdity of *A*, but not vice versa." Fortunately the ideas of Brouwer have been interpreted in English (Dresden [**19**], [**20**]). The position of the intuitionists is radical enough to demand the revision of a large part of mathematics.¶

Poincaré was an earlier and livelier intuitionist, who emphasized the importance of mathematical induction as an extra-logical element in the foundation of mathematics.** Weyl gives in [**57**] (cf. also [**58**]) a profound and stimulating discussion of these ideas. He emphasizes rightly the importance of constructive proofs for existence theorems. If, on this view, the proof of an existence statement can be accomplished only by exhibiting the object supposed to exist, then it follows that some existence statements are neither true nor false.

There are available many semi-popular discussions of intuitionism as compared with other schools of thought in the foundations of mathematics.†† One warning on these discussions is necessary: It was once fashionable to discuss just three schools; intuitionism, formalism, logistic. This neat trichotomy is no longer adequate. It fails to take into account the interaction of intuitionism and formalism and the recent work of Church, Curry, Carnap, Quine and others (Church [**15**]). An up-to-date discussion appears in Gentzen [**26**], [**27**].

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* Stone [**52**], Huntington [**36**], [**37**], Bernstein [**7**], and other references given in these papers. Sheffer [**51**] is important.

† Lewis-Langford [**42,** chapter 3, part 2].

‡ Lewis-Langford [**42,** chapter 4].

§ Frink [**25**], Lewis-Langford [**42,** chapter 7].

‖ Easier than most is Brouwer [**10**]; cf. also the bibliography and general discussion in Heyting [**31**].

¶ Considerable progress on this revision has been made. Cf. Heyting [**31,** part 1].

** [**46,** Science and Hypothesis, chapter 11; Science and Method, chapters 3, 4, and 5.]

†† Fraenkel [**24,** §§14, 15, 18]; Hedrick [**30**]; Bell [**2,** pp. 575–579]; Black [**9,** pp. 9–11, 169–211]; Frink [**25**]; Menger [**44**]; Dubislav [**21**].

9. Geometry and logic. An amusing geometrical interpretation of intuitionism has recently been found.* It has long been customary to treat Boolean algebra by Venn diagrams.† A class A is pictured as the set of points inside a circle; the complement of the class A (the negation of the corresponding proposition) then consists of the points outside the circle. If the boundary is neglected, A and not $-A$ together constitute the whole plane—but if the boundary is considered there are points which are neither A nor not $-A$, exactly as in intuitionist logic! This has been developed technically for very general spaces, but the underlying ideas could be presented to a club in a simple, attractive form, using ordinary Euclidean geometry.

Quantum mechanics has also led to the introduction of novel logics (Birkhoff-von Neumann [**8**]).

10. Consistency proofs. A fundamental problem in the foundation of mathematics may be stated thus: Can the postulates for all the ordinary parts of mathematics be chosen so that all the ordinary paradoxes are avoided and so that we can also prove that no new contradictions or paradoxes could arise? Hilbert has attempted to use symbolic logic to furnish such a "consistency proof." An early discussion of his ideas appears in [**32**]. Only for limited parts of mathematics has such a proof been obtained.‡ One especially important idea has developed from these investigations. One wishes to consider a possible proof which leads to an inconsistency within a system of mathematics; to do this one must go outside the given system and talk *about* it. This "outside" metamathematics or syntax is discussed in Bernays [**6**], Carnap [**12**], Hilbert-Bernays [**34**]. A clear understanding of this "syntax" is necessary in any modern system of logic.

11. Gödel's theorem. The most influential result in modern symbolic logic is the Theorem of Gödel concerning statements which can be neither proven nor disproven in a given logical system. It raises enormous difficulties for the Hilbert program, for one consequence is the impossibility of proving the consistency of most systems by the methods of proof used in those systems.§ The basic idea of Gödel's proof can be presented to a club that is not too immature, for it depends on a suitable use of one of the paradoxes. A good outline is given in the introduction to Gödel's original paper [**28**]; the details may be found there or in Gödel [**29**] or Carnap [**12**, §§35 and 36]. There is a general discussion of the Theorem in Menger [**44**]. The difficulty in understanding this Theorem is well repaid by the importance of the result.

12. Miscellaneous topics. *History of logic*: Jørgensen [**39**, volume 1]; Lewis [**41**, chapter 1]; Lewis-Langford [**42**, chapter 1]; Bell [**2**]. *Transfinite numbers*: Russell [**50**]; Russell [**48**, chapters 11, 13, 43]; Bell [**2**]. *Strict implication*: Lewis-Langford [**42**, chapter 6]; Lewis [**41**]; Huntington [**37**]; Feys [**23**]. The role of "strict implication" is subjected to a penetrating analysis in Carnap [**12,** §§69–70]. *Applications of logic to philosophical problems* Carnap [**12**]. *Calculus of Relations*: Whitehead-Russell [**59**]. *Nature of Deductive Systems*: [**13,** chapters 9 and 10].

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* Stone [**54**], Tarski [**56**].

† Lewis-Langford [**42**, chapter 3]; Eaton [**22**, chapter 3]; Lewis [**41**, chapter 3, part 1].

‡ Details in Bernays [**5**, pp. 56–100]; Hilbert-Bernays [**34**]; Summaries in Mac Lane [**43**]; Heyting [**31**, part 2]; Black [**9**, pp. 147–169]; Weyl [**57**, §10]; Gentzen [**27**].

§ See, however, the discussion in Gentzen [**26**], [**27**].

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CLUB REPORT

Pi Mu Epsilon, New York University

The chapter held regular meetings during the college year. Papers were presented on Euclid's parallel postulate; Bishop Berkeley's criticism of differentiation; Dimension theory; Function spaces. The guest speaker of the year was Professor W. V. Quine of Harvard University who spoke in February on The theory of logical types. The fifth annual Pi Mu Epsilon Interscholastic Mathematics Contest was held on April 30, 1938. There were 114 high schools represented, with a total of 430 contestants. The highest score was made by the Abraham Lincoln High School of Brooklyn, N. Y., which was awarded the large cup. The team placing second was from Boys High School, Brooklyn, N. Y. Four smaller cups were awarded to the teams making the highest score in their section. Medals were awarded to the three contestants who made the highest individual scores. The chapter also conducted two mathematics contests for undergraduates, one for freshmen and the other for upper-classmen. In each case an examination was given, and a prize of $20 was offered for the best paper in each contest.

Director, Professor F. W. John; Vice-Director, H. I. Zagor; Secretary, Frieda Agin; Treasurer, Naomi Rosenstein, Permanent Secretary, J. H. Moss.

"THE MATHEMATICAL SAGA OF LINNIE R. E. QUASHUN."

Below are words which may be used to fill in spaces in the story printed last month, pp. 234–235.

1. regions
2. sum
3. planes
4. operations
5. slipstick
6. radicals
7. elliptical
8. high
9. quotient
10. line
11. proportion
12. figure
13. slope
14. circumference
15. infinitesimal
16. foot
17. nine
18. thirty
19. six
20. curves
21. mean
22. integrate
23. difference
24. root
25. table
26. unknown
27. proof
28. tangent
29. less
30. plane
31. indeterminate
32. height
33. satisfied
34. increment
35. added
36. function
37. normal
38. subnormal
39. circles
40. limit
41. inclined
42. perspective
43. whole
44. system
45. rational
46. index
47. exponent
48. positive
49. multiply
50. parallel
51. spaces
52. lemniscate
53. times
54. demonstration
55. period
56. coördinate
57. greater
58. pi
59. hypotenuse
60. table
61. ratio
62. locus
63. one
64. logarithm
65. root
66. power
67. compute
68. mechanics
69. differential
70. reduced
71. rhombus
72. series
73. maximum
74. group
75. base
76. altitude
77. problem
78. solution
79. irrational
80. continuity
81. secant
82. terminate
83. square
84. rate
85. properties
86. center
87. complementary
88. minimum
89. revolved
90. coincided
91. loci
92. remainder
93. vertical
94. horizontal
95. exercise.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 378. *Proposed by J. L. Brenner, University of Minnesota.*

Find the number of integral values of B which make B^2+m a perfect square, for any given, fixed, integer m. If so, then, $B^2+m = (B+S)^2 = B^2+2B$

E 379. *Proposed by W. E. Buker, Pittsburgh Public Schools.* $m = 2BS+$

Find a trapezoid whose sides, altitude, diagonals and area are rational. $B =$

E 380. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

If the radius of a circle is any odd prime, p, there are just two different primitive Pythagorean triangles circumscriptible about that circle. Show that, for each such pair of triangles:

(A) their shortest sides differ by one;

(B) their hypothenuses exceed their corresponding longer legs by one and by two respectively;

(C) the sum of their perimeters is six times a perfect square;

(D) as p increases without limit, the ratio of their least angles approaches the limit 2;

(E) as p increases without limit, the ratio of their areas approaches the limit 2; and finally,

(F) the smaller triangle can always be placed inside the larger, so as not to touch it.

E 381. *Proposed by W. B. Clarke, San Jose, California.*

Show how to construct a square with one corner on each of four generally placed straight lines in a plane. How many solutions are there in general? What constitute special cases? What happens if the lines are placed askew in space?

E 382. *Proposed by Virgil Claudian, Bucharest, Roumania.*

A, B and C are three fixed points in space. (S) is a sphere fixed in space, bearing A and B, but not C, on its surface. P is a point which moves over the surface of (S). (Q) is the sphere determined by A, B, C and P. T is the plane tangent to Q at C. M is the plane ABP. Planes M and T intersect in the line l. Determine the locus of l.

SOLUTIONS

E 314 [1938, 47]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Inst.*

Find the locus of the center of a variable sphere which cuts two (or three) fixed planes in circles of constant size.

Solution by G. A. Yanosik, New York University.

Let the two fixed planes be $y = \pm mx$. Let the center of the variable sphere be (X, Y, Z). Let the radii of the circles of section with the two fixed planes be q and r respectively.

Since all radii of the sphere at any instant are equal, we have

$$q^2 + (mX - Y)^2/(1 + m^2) = r^2 + (mX + Y)^2/(1 + m^2),$$

which reduces to $4mXY = (q^2 - r^2)(1 + m^2)$, a hyperbolic cylinder whose generators are parallel to the line of intersection of the two fixed planes.

If three fixed planes are given, the locus of (X, Y, Z) is obviously a space curve, the common curve of intersection of three hyperbolic cylinders with elements respectively parallel to the three lines of intersection of the fixed planes taken in pairs.

Also solved by L. E. Malvern and F. L. Wilmer.

E 338 [1938, 320]. *Proposed by C. W. Trigg, Los Angeles City College.*

There are only two numbers which are permutations of the ten digits, which are also the products of three consecutive integers. Find them and show that there are no others.

Solution by W. E. Buker, Pittsburgh Public Schools.

Let the consecutive integers be $n-1$, n and $n+1$, so that their product is $n^3 - n$. It is readily seen that n lies between 1000 and 2147, and differs by not more than 1 from a multiple of 9. Using a table of cubes to seek values of n which satisfy these two conditions, we reject cubes showing any duplication of digits in the first six digits at the left. We subtract n from the remaining eligible cubes and again reject all cases where any digit appears more than once. The only two cases remaining are

$$1267 \cdot 1268 \cdot 1269 = 2,038,719,564$$

and

$$1332 \cdot 1333 \cdot 1334 = 2,368,591,704.$$

Also solved by the proposer.

E 339 [1938, 478]. *Proposed by V. Thébault, Le Mans, France.*

Consider all triangles inscribed in a given circle on a fixed chord as base, and determine the locus of the feet of the interior and exterior base-angle bisectors. Thus show that if two interior angle bisectors of a triangle are equal, the triangle is isosceles.

Solution by the proposer.

If side BC and angle A of triangle ABC remain constant, the feet of the bisectors of angles B and C generate two strophoids which are symmetrically placed with respect to the perpendicular bisector of BC. This follows at once from the definition of a strophoid. Hence if triangle ABC has the two interior angle bisectors, BD and CE, equal, the points D and E are symmetrically placed with respect to the perpendicular bisector of BC, DE is parallel to BC, and triangle ABC is isosceles.

E 340 [1938, 478]. *Proposed by E. C. Kennedy, Texas College of Arts and Industries.*

If B is a positive integer, what rational values may be assumed by

$$R=\sqrt{\left(\frac{B}{7}\right)^2+5038}.$$

Solution by E. P. Starke, Rutgers University.

We have at once $(7R)^2=B^2+49\cdot 5038$. Since $7R$ is to be rational and has an integral square, it is an integer. Now $7R-B$ and $7R+B$ are of like parity, and since their product is even, they must both be even. But that requires their product to be a multiple of four, which it is not. Hence we have a contradiction, and there is no integer value of B which makes R rational.

However, if we remove the requirement that B be an integer, there are infinitely many rational values for B and R, since we may let $7R-B$ equal any rational value, K, and let $7R+B=49\cdot 5038/K$. Then $B=(49\cdot 5038-K^2)/2K$ and $R=(49\cdot 5038+K^2)/14K$.

Editorial Note. As originally proposed in manuscript, this problem had 503 instead of 5038, which led by the methods shown above to the three solutions, $B=227$, 1757 and 12323, and $R=276/7$, 252 and 12324/7. Although the error in transcription has not been traced to its origin, appropriate apologies are hereby tendered to all concerned.

Also solved by J. L. Brenner, Fred Discepoli, F. Olson, C. W. Trigg and the proposer.

E 341 [1938, 478]. *Proposed by J. H. Edmonston, Washington, D. C.*

A triangular octahedron (one with all faces triangular) may be regarded as a space analogue of a plane quadrilateral. On this basis, state and prove a space-analogue of the theorem that the midpoints of the sides of any plane quadrilateral are the vertices of a parallelogram.

Solution by C. W. Trigg, Los Angeles City College.

THEOREM. *The centroids of the faces of a triangular octahedron are the vertices of a parallelopiped.*

Let the edges issuing from a vertex, V, of a triangular octahedron be VA, VB, VC and VD. From B and D, and in the faces determined by these four edges, draw the medians to VA and VC. On these medians locate the centroids, E, F, G and H, of the faces. Draw BD, EF, FG and HE. Then $EH=BD/3=FG$, EH and FG are parallel to BD, so that $EFGH$ is a parallelogram. Repeating this process at the other five vertices, we show the faces of the figure determined by the eight face-centroids to be all parallelograms, so that the figure is a parallelopiped.

Also solved by W. E. Buker, E. P. Starke and the proposer.

E 342 [1938, 478]. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Show how to draw three circles with radii a, b and c, and common tangents d, e and f.

Solution by W. B. Clarke, San Jose, California.

Since the problem does not specify whether the tangents are internal or external, both cases must be considered. The distance between the centers of two circles of radii b and c, with a tangent of length d, is obviously

$$\sqrt{d^2 + (b \pm c)^2}$$

where the alternative signs correspond to d being an internal or external tangent respectively. Since such a length is readily constructible, the centers of the desired circles become the vertices of a triangle with known sides, and since the respective radii are also known, the construction is obvious.

While there are in general eight solutions, correspodning to the different possible combinations of internal and external tangents, any number of these may disappear in special cases, when the corresponding triangles become outlawed through having the length of one side exceed the sum of the lengths of the other two.

Also solved by E. R. Heineman, D. L. MacKay, Joseph Milkman and the proposer.

E 343 [1938, 478]. *Proposed by H. E. Stelson, Kent State University.*

Prove that $ax^2+2hxy+by^2+2gx+2fy+c$ is the product of two factors, linear and rational in x and y, provided $[h+(h^2-ab)^{1/2}][f+(f^2-bc)^{1/2}]-b[g+(g^2-ac)^{1/2}]=0$, and that in this case, $abc+2fgh-af^2-bg^2-ch^2=0$.

Solution by W. E. Buker, Pittsburgh Public Schools.

Assume factors $c(lx+my+1)(rx+sy+1)$, whose product is $c[lrx^2+(ls+mr)xy+msy^2+(l+r)x+(m+s)y+1]$. Equating the coefficients to those in the given expression, we get

(A) $clr=a$, $cms=b$, $c(ls+mr)=2h$, $c(l+r)=2g$, $c(m+s)=2f$.

Solving these equations for l gives $l=a/cr=(2g-cr)/c=(2h-cmr)/cs$. This enables us to eliminate l from (A), getting

(B) $cms=b$, $2gr-cr^2=a$, $2hr-cmr^2=as$, $cm+cs=2f$.

This time we solve for m where possible, getting $m=b/cs=(2hr-as)/cr^2=(2f-cs)/c$, which lets us eliminate m, getting

(C) $cr^2-2gr+a=0$, $as^2-2hrs+br^2=0$, $cs^2-2fs+b=0$.

The next step is to solve the first and third of equations (C) for r and s, and substitute in the second, getting

$$\text{(D)}\quad a[f\pm(f^2-bc)^{1/2}]^2-2h[f\pm(f^2-bc)^{1/2}][g\pm(g^2-ac)^{1/2}]+b[g\pm(g^2-ac)^{1/2}]^2 = 0.$$

For all the ambiguous signs we use only the plus, and solve the above equation as a quadratic in $[g+(g^2-ac)^{1/2}]$, arriving at

$$\text{(E)}\quad [g+(g^2-ac)^{1/2}]=[h+(h^2-ab)^{1/2}][f+(f^2-bc)^{1/2}]/b$$

which is the required relationship.

To verify the latter part of the problem, it is sufficient to notice that the values for a, b, f, g and h, as given in (A), reduce $abc+2fgh-af^2-bg^2-ch^2=0$ to an identity.

Also solved by E. P. Starke and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3848 [1937, 667]. *Proposed by P. Erdös, Budapest, Hungary.*

(*Correction.*) Let O be an arbitrary point in the interior of the triangle ABC, and let A', B', C' be the points in which AO cuts BC, etc. If $AA'\geqq BB'$ and $AA'\geqq CC'$, then $AA'\geqq OA'+OB'+OC'$, where the equality holds only if $AA'=BB'=CC'$.

3913. *Proposed by W. B. Campbell, Drexel Institute of Technology.*

If $f(x)=e^{kx}\tan x$, $k\geqq 0$, discuss the effect of the value of k on the location of maximum, minimum, and inflection points.

3914. *Proposed by W. B. Campbell, Drexel Institute of Technology.*

A dog is tied to a rope of length L, which is fastened on the other side of a smooth topped fence of height h at a point a units from the top, $L>h+a$. Discuss the shape and dimensions of the region over which he can roam; and find its area.

3915. *Proposed by V. Thébault, Le Mans, France.*

Two numbers have each eight figures the last of which is seven in each one; and the first eight figures of their product are identical as well as the last eight. Find the two numbers.

3916. *Proposed by V. Thébault, Le Mans, France.*

Find a sequence of three figures such that if the sequence is regarded first as a number in a system with a certain unknown base, and then as a number in the duodecimal system (base 12), the value of the number in the second case is one half of that in the first. Show that there are only two such unknown bases for the given problem.

3917. *Proposed by R. A. Johnson, Brooklyn College.*

Determine the values of n for which the sum of the squares of n positive integers in arithmetic progression is the square of an integer, and show how to determine such progressions.

SOLUTIONS

3822 [1937, 179]. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

In a triangle, the inscribed conics passing through the intersections of the circum-and conjugate circles are tangent to the nine-point circle.

Solution by Otto J. Ramler, Catholic University of America.

In an article entitled "A geometrical proof of Professor Morley's extension of Feuerbach's Theorem" (Proceedings Edinburgh Mathematical Society, 1920, vol. 38, p. 2) H. W. Richmond develops the following theorems upon which we base our solution:

1) If the three unknown common tangents of two line cubics which touch the six sides of the complete quadrangle $OPQR$ form a triangle XYZ, the seven points $OPQRXYZ$ lie on a conic and conversely,

2) If the vertices of a triangle XYZ lie on a conic with the points $OPQR$ then all the line cubics touching OP, OQ, OR, QR, RP, PQ and two sides of XYZ also touch the third side.

3) If L, M, N are the points of contact of YZ, ZX, XY with a line cubic touching the six lines mentioned above, then XZ, ZX, XY are also tangent to a conic at L, M, N and hence XL, YM, ZN are concurrent.

4) If the positions of ZX, ZY, L and M are known, those of XY and N are readily found.

5) For the system Γ_3 of line cubics which touch the six lines OP, OQ, OR, QR, RP, PQ and which pass through two given points L and M there is a relation between the tangents LZ and MZ. Hence there is a locus for Z.

6) For the system of conics through $OPQR$ and conjugate with respect to L and M the locus of N, the pole of LM as to these conics is another conic through L and M and circumscribed to the diagonal triangle ABC of $OPQR$.

7) When two adjacent Γ_3's touching the six lines OP, OQ, OR, QR, RP, PQ and passing through L and M approach coincidence, their nine common tangents are ultimately

i) the six lines OP, OQ, OR, QR, RP, PQ

ii) the two lines LYZ, MXZ through L and M, the points of contact being L and M;

iii) a line XNY touching the conic $LMABC$ at N, the point of contact with the Γ_3 also being N.

The envelope of the cubics is thus seen to be the points L and M and the conic $LMABC$.

Now let L, M be the intersections of the circumscribed circle PQR and the conjugate circle of PQR. It is known that the nine-point circle also belongs to this pencil. It can also be readily proved that all conics through O, P, Q, R, where O is the centroid of PQR, are conjugate to L and M, and that the locus of N, the pole of LM as to the pencil of conics is the nine-point circle. Hence the envelope of the system of line cubics touching OP, OQ, OR, QR, RP, PQ and passing through L and M is the nine-point circle ABC and the points L, M, where ABC is the medial triangle of PQR. Now in the system of line cubics there are four degenerate cubics each consisting of a vertex of the quadrangle $OPQR$, and the line conic touching the other three lines not passing through that vertex. Hence there are sixteen conics through the intersections L, M of the circumcircle and the conjugate circle and inscribed to the four triangles formed by the four points O, P, Q, R, and each of these conics touches the nine-point circle.

3823 [1937, 179]. *Proposed by J. R. Musselman, Western Reserve University.*

Given any triangle $T_1T_2T_3$, let us denote by T_1', T_2', T_3' the reflections of T_1, T_2, and T_3 in any diameter of the circumcircle. The lines through T_i perpendicular to $T_j'T_k'$ meet at a point R_t, similarly the lines through T_i' perpendicular to T_jT_k meet at a point M_t. It is known that the points R_t and M_t also lie on the circumcircle and are symmetric with respect to the above mentioned diameter. Given three sets of three points T_i, V_i, W_i on the same circle, locate with reference to some diameter the points R_t, R_v, R_w. Similarly for the triangles $T_1V_1W_1$, $T_2V_2W_2$ and $T_3V_3W_3$ find R_1, R_2 and R_3. Show that the triangles $R_1R_2R_3$ and $R_tR_vR_w$ have the same R point. A like theorem can be stated for the M point.

Solution by C. E. Springer, University of Oklahoma.

Let $x^2+y^2=a^2$ be the common circle of the sets of points T_i, V_i, W_i with the diameter of reflection along the x-axis. If the coordinates of T_i, V_i, W_i are $(a\cos\alpha_i, a\sin\alpha_i)$, $(a\cos\beta_i, a\sin\beta_i)$ and $(a\cos\gamma_i, a\sin\gamma_i)$, respectively, then the line through T_1 perpendicular to T_2', T_3' is

$$x\sin\tfrac{1}{2}(\alpha_2+\alpha_3)+y\cos\tfrac{1}{2}(\alpha_2+\alpha_3)=a\sin\tfrac{1}{2}(2\alpha_1+\alpha_2+\alpha_3)$$

and the line through T_2 perpendicular to $T_3'T_1'$ is

$$x\sin\tfrac{1}{2}(\alpha_3+\alpha_1)+y\cos\tfrac{1}{2}(\alpha_3+\alpha_1)=a\sin\tfrac{1}{2}(2\alpha_2+\alpha_3+\alpha_1).$$

The point R_t common to these lines has coördinates $[-a\cos(\alpha_1+\alpha_2+\alpha_3)$, $a\sin(\alpha_1+\alpha_2+\alpha_3)]$. Similarly, we find $R_v[-a\cos(\beta_1+\beta_2+\beta_3), a\sin(\beta_1+\beta_2+\beta_3)]$ and $R_w[-a\cos(\gamma_1+\gamma_2+\gamma_3), a\sin(\gamma_1+\gamma_2+\gamma_3)]$. Likewise, the coördinates of the R points $R_i(i=1, 2, 3)$ are given by $[-a\cos(\alpha_i+\beta_i+\gamma_i), a\sin(\alpha_i+\beta_i+\gamma_i)]$, $(i=1, 2, 3)$, and the R point of the triangle $R_1R_2R_3$ has coördinates $(-a\cos\theta$, $a\sin\theta]$ where $\theta=\alpha_1+\beta_1+\gamma_1+\alpha_2+\beta_2+\gamma_2+\alpha_3+\beta_3+\gamma_3$. The R point of $R_tR_vR_w$ is also $[-a\cos\theta, a\sin\theta]$. By symmetry a like result holds for the M point.

Editorial Note. It is possible to avoid the trigonometric reductions. For the T triangle the angles of its vertices are α_1, α_2, α_3, as in the above solution; and we may take for the vertices of the T' triangle the angles $-\alpha_1$, $-\alpha_2$, $-\alpha_3$. Let the perpendicular from T_i to $T_j'T_k'$, where i, j, k are distinct, cut the circle again in R_i with the angle θ_i. The perpendicular from O, the center of the circumcircle (O), to $T_j'T_k'$ has the angle $-(\alpha_j+\alpha_k)/2+m_1\pi$, while the perpendicular from O to T_iR_i has the angle $(\theta_i+\alpha_i)/2+m_2\pi$, where m_1, m_2 are positive or negative integers. Since the above two perpendiculars are also perpendicular, we have $(\theta_i+\alpha_i+\alpha_j+\alpha_k)/2+(m_2-m_1)\pi=(2k+1)\pi/2$, or $\theta_i=(2n+1)\pi-\sigma_\alpha$, $\sigma_\alpha=\alpha_1+\alpha_2+\alpha_3$. It suffices to take $\theta_i=\pi-\sigma_\alpha$. This proves that the three perpendiculars for $i=1, 2, 3$, meet in the same point R_t on (O) given by the angle $\pi-\sigma_\alpha$. The theorem for M_t is then obvious by reflection. We obtain special cases by making two vertices of the T triangle coincide.

Now let there be nine points on the circle (O) with the angles $\alpha_i, i=1, 2, \cdots, 9$, which we take as distinct although this is not necessary, Separate the nine points into three sets of three points. We then have three triangles α_{i_1}, α_{i_2}, α_{i_3}; α_{j_1}, α_{j_2}, α_{j_3}; α_{k_1}, α_{k_2}, α_{k_3}. The first triangle gives the point R_i, $\theta_i=\pi-\sigma_i$; the second, R_j, $\theta_j=\pi-\sigma_j$; the third, R_k, $\theta_k=\pi-\sigma_k$. The triangle $R_iR_jR_k$ gives a point R with the angle $\theta=\pi-(\pi-\sigma_i)-(\pi-\sigma_j)-(\pi-\sigma_k)=\Sigma\alpha_i-2\pi$. Hence we take $\theta=\Sigma\alpha_i$. This proves that the 280 such systems lead to the same point R. Similarly for the M point.

3824 [1937, 251]. *Proposed by F. A. Lewis, University of Alabama.*

Determine the roots of the characteristic equation of the matrix

$$V=(v_{rc})=(\epsilon^{(r-1)(c-1)})\qquad\text{where}\qquad \epsilon=e^{2\pi i/n}.$$

Solution by J. S. Frame, Brown University.

If V is the matrix $\epsilon^{(r-1)(c-1)}$, and $\epsilon=e^{2\pi i/n}$ then

$$V^2=\left(\sum_{s=1}^{n}\epsilon^{(s-1)(r+c-2)}\right)=\begin{pmatrix} n & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & n\\ 0 & 0 & 0 & \cdots & n & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & n & \cdots & 0 & 0\\ 0 & n & 0 & \cdots & 0 & 0\end{pmatrix}.$$

The roots of $(V^2-\lambda^2I)=0$ are $\pm n$, the sum being $2n$ if n is even, and n if n is odd. Hence for $[n/2]+1$ roots $\lambda^2=n$, and for the rest $\lambda^2=-n$. The roots of the given matrix V are $\sqrt{n}$, $i\sqrt{n}$, $-\sqrt{n}$, $-i\sqrt{n}$, and if these occur with the respective multiplicities m_0, m_1, m_2, and m_3, then we have

$$m_0+m_1+m_2+m_3=n,\qquad m_0+m_2=[n/2]+1.$$

According to the theory of Gauss sums, the trace of the matrix V has the value $(1+i)\sqrt{n}$, $\sqrt{n}$, 0, $i\sqrt{n}$, according as $n\equiv 0, 1, 2, 3 \pmod 4$. Since this is also $(m_0+im_1-m_2-im_3)\sqrt{n}$, we find

$$\begin{aligned} m_0 - m_2 &= 1 \quad \text{when} \quad n = 4k \text{ or } 4k+1 \\ &= 0 \quad \text{when} \quad n = 4k+2 \text{ or } 4k+3, \\ m_1 - m_3 &= 1 \quad \text{when} \quad n = 4k \text{ or } 4k+3 \\ &= 0 \quad \text{when} \quad n = 4k+1 \text{ or } 4k+2. \end{aligned}$$

It follows that the multiplicities are given in all cases by

$$m_0 = \left[\frac{n}{4}\right] + 1, \quad m_1 = \left[\frac{n+1}{4}\right], \quad m_2 = \left[\frac{n+2}{4}\right], \quad m_3 = \left[\frac{n+3}{4}\right] - 1.$$

3825 [1937, 251]. *Proposed by H. T. R. Aude, Colgate University.*

A number, written in the scale of 10, has the digit d ($d=2, 3, \cdots, 9$) in the position on the extreme right. A second number is formed by moving the digit d to the position on the extreme left. If the second number is d times the first, find the least number of digits n of the numbers for the various values of d. Also find these eight numbers, or obtain a formula for them.

Solution by Foster Brooks, Kent State University, Kent, Ohio.

The eight numbers required may be obtained by direct multiplication. For instance if $d=2$, a simple multiplication problem is formed with the multiplicand blank except for the final digit 2, and with multiplier 2. Thus 4 is the last digit in the product, and hence next-to-the-last in the multiplicand. As each digit of the product is found it is transferred to the multiplicand and the process is continued until the digit 2 appears in the product with nothing to carry.

The results are given in the following table:

d	n	number
2	18	105, 263, 157, 894, 736, 842.
3	28	1, 034, 482, 758, 620, 689, 655, 172, 413, 793.
4	6	102, 564.
5	42	102, 040, 816, 326, 530, 612, 244, 897, 959, 183, 673, 469, 387, 755.
6	58	1, 016, 949, 152, 542, 372, 881, 355, 932, 203, 389, 830, 508, 474, 576, 271, 186, 440, 677, 966.
7	22	1, 014, 492, 753, 623, 188, 405, 797.
8	13	1, 012, 658, 227, 848.
9	44	10, 112, 359, 550, 561, 797, 752, 808, 988, 764, 044, 943, 820, 224, 719.

Note. The existence of solutions of this problem does not depend upon the use of the decimal scale of notation, or upon the fact that the second number is as many times the first as the digit moved.

Suppose it is required to find a number which when written in the system with radix r ends in the digit d, and such that if this digit is moved to the position on the extreme left the new number is m ($m\leqq d$) times the original. If n

denotes the number of digits in the original number, and if x is the number formed by the first $n-1$ digits of the original number, then $m(rx+d)=r^{n-1}d+x$, or,

$$x = \frac{r^{n-1}d - dm}{rm - 1}.$$

Here x will be integral if, and only if,

$$r^{n-1}d - dm \equiv 0 \ (\text{mod } rm - 1),$$

and hence if, and only if,

$$r^{n-1} \equiv m \left(\text{mod} \frac{rm - 1}{g}\right),$$

where g is the greatest common divisor of $rm-1$ and d. Since r is prime to $rm-1$, and hence to $(rm-1)/g$, the equation just above is equivalent to

$$r^n \equiv rm \equiv 1 \left(\text{mod} \frac{rm - 1}{g}\right),$$

which has one solution $n=\phi[(rm-1)/g]$. Thus the least value for n is either $\phi[(rm-1)/g]$ or one of its divisors.

Solved also by M. T. Bird, W. E. Buker, J. E. Davis, Daniel Finkel, E. P. Starke, C. W. Trigg, and the proposer.

Editorial Note. Buker, Davis, and Starke computed the table for the eight numbers by the multiplication rule in the above solution. Trigg reversed this process by dividing d into a dividend whose first three digits at the left are $d10$; the fourth digit of the dividend is the third digit of the quotient. This is continued until d appears in the quotient without a remainder. Bird obtained the formula $(10d-1)p=d(10^n-1)$ or

$$10^{n-1} + (10^{n-1} - 1)/(10d - 1) = p + (d - 1)/(10d - 1),$$

where p is the original number with the last digit d. Hence it suffices to divide $999 \cdots 9$ by $10d-1$ until the remainder $d-1$ appears and to place 10 before the quotient. Finkel also used this method. Before computing his table, Starke showed that n, the least number of digits, must be the least integer not zero which satisfies $10^n\equiv 1$, mod $(10d-1)$, where n must be a divisor of $\phi(10d-1)$; and then he computed, independently of the table, the eight values of n corresponding to the eight values of d. He also stated that one method for determining the exponent of 10 mod 79 is analogous to the multiplication rule above. Thus, for $10^n\equiv 1$, mod 79, $10^n\equiv 1+79=80$; hence $10^{n-1}\equiv 8\equiv 8+8\cdot 79=640$; hence $10^{n-2}\equiv 64\equiv 64+4\cdot 79=380$; hence $10^{n-3}\equiv 38$. We continue in this way until we have $10^0\equiv 1$. The proposer showed that the required sequence of digits constitutes the recurring part of the decimal form of $d/(10d-1)$. The number of digits, n, is the same as the least number of 9's in succession forming a number divisible by $(10d-1)$. These facts result at once from the first of Bird's equations above. He also mentioned the generalization to numbers in any scale s.

NEWS AND NOTICES

Readers are invided to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Professor Karl Menger of Notre Dame University has announced "a little publication to be issued annually *Reports of a Mathematical Colloquium*, second series, issue 1, a continuation of the *Ergebnisse eines mathematischen Kolloquiums*" which he formerly published in Vienna.

On April 28 mathematicians of the University of Wisconsin were guests of the mathematics department of Northwestern University. Dr. Ernst Hellinger, recently appointed lecturer at the latter institution, gave an address on "Types of Integrals." At a dinner meeting the guests of honor were Dean C. S. Slichter, Wisconsin (B.S., Northwestern, 1885), and Dean T. F. Holgate, Northwestern (A.B., Toronto, 1884). By report "the home team won a tennis aftermath."

Dr. W. D. Cairns, professor of mathematics at Oberlin College, will retire in June 1939 after forty years of service there. His duties as secretary-treasurer of the Mathematical Association of America will continue as heretofore, no change of address being involved.

At Princeton University Salomon Bochner, H. F. Bohnenblust, and Alonzo Church, have been promoted from assistant professorships to associate professorships.

Dr. N. E. Steenrod of Princeton has been appointed to an assistant professorship at the University of Chicago.

Dr. R. W. Wagner of the University of Wisconsin has been appointed instructor in mathematics at Oberlin College for the next academic year.

Professor N. C. Grimes of Grove City College died November 12, 1938.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The following results of the second annual William Lowell Putnam Mathematical Competition held March 4, 1939, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to Association officials and to the reader only by number up to the time of this announcement.

The first prize, five hundred dollars, is awarded to the department of mathematics of Brooklyn College, Brooklyn, New York. The members of the team were Richard Bellman, Abraham Hillman, Bernard Sherman; to each of these is awarded a prize of fifty dollars.

The second prize, three hundred dollars, is awarded to the department of mathematics at Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were R. P. Feynman, J. W. Follin, Jr., H. E. Singleton; to each of these is awarded a prize of thirty dollars.

The third prize, two hundred dollars, is awarded to the department of mathematics of Mississippi Woman's College, Hattiesburg, Mississippi. The members of the team were Nina P. Byrd, Mary E. Fancher, Ethel L. Tate; to each of these is awarded a prize of twenty dollars.

The five persons ranking highest in the examination, arranged in alphabetical order, were R. P. Feynman, Massachusetts Institute of Technology; Abraham Hillman, Brooklyn College; E. L. Kaplan, Carnegie Institute of Technology; William Nierenberg, College of the City of New York; Bernard Sherman, Brooklyn College. Each of these will receive a prize of fifty dollars. The order of the names in this list has no relation to their rank in the examination.

The following teams won honorable mention: department of mathematics, College of the City of New York, members of the team being Herbert Mintzer, William Nierenberg, Harry Soodak; department of mathematics, Cooper Union Institute of Technology New York, members of the team being Theodore Berlin, Benjamin Lax, Samuel Manson; department of mathematics, University of California, Berkeley, members of the team being W. M. Kincaid, C. W. Lippmann, S. A. Schaaf.

Seven individuals are given honorable mention because of ties in the scores in this group. The seven names, arranged alphabetically, are: Richmond Albert, Brooklyn College; Richard Bellman, Brooklyn College; Theodore Berlin, Cooper Union Institute of Technology; Benjamin Lax, Cooper Union Institute of Technology; C. W. Lippmann, University of California; M. J. Norris, College of St. Thomas; T. S. Schreiber, Johns Hopkins University.

The order of the names in both lists for honorable mention has no relation to their rank in the examination.

The following is a list of all colleges and universities which entered teams in the Competition. (This list is arranged alphabetically, and the order of the names here has no relation to the rank of the teams in the examination.) Antioch College, Bowdoin College, Brooklyn College, Carleton College, College of St. Catherine, College of the City of New York, Columbia University, Cooper Union Institute of Technology, Cornell College (Mt. Vernon, Iowa), Harvard University, Heidelberg College, Iowa State College (Ames), Kent State University, Knox College, Lafayette College, Lehigh University, Massachusetts Institute of Technology, Michigan College of Mining and Technology, Middlebury College, Mississippi Woman's College, Mount Saint Scholastica College, Mundelein College, Washington Square College of New York University, Park College, Pomona College, Queen's University, Rutgers University, Union College (Schenectady, New York), the universities of British Columbia, Buffalo, California. California at Los Angeles, Tampa, Virginia, Washington, Wisconsin; Virginia Military Institute, Washington and Jefferson College, Webb Institute of Naval Architecture, Western Reserve University.

In addition to these forty-one teams, there were seventy-seven individual contestants from these and twenty-eight other institutions, making a total of two hundred individuals representing sixty-nine institutions.

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